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Research Article

INTEGER SOLUTION OF THE NON-HOMOGENEOUS BIQUADRATIC EQUATION WITH SIX UNKNOWNS $x^4 - y^4 - 8zw = 2T(P^2 + 1)$

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ABSTRACT

We obtain infinitely many non-zero integer sextuples (x, y, z, w, P, T) satisfying the biquadratic equation with six unknowns $x^4 - y^4 - 8zw = 2T(P^2 + 1)$. Various interesting properties among values of x, y, z, w, P and T are presented.

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1. INTRODUCTION

The theory of diophantine equations offers a rich variety of fascinating problems. In particular, biquadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity (**Dickson, 1952, Mordell, 1969, Carmichael, 1959, Telang, 1996, Nigel and Smart 1999**). In this context, one may refer (**Gopalan *et al.*, 2009,2010,a,b,c,2012a,b; Meena et al., 2014**) for various problems on the biquadratic diophantine equations with four variables and (**Gopalan *et al.*, 2009,2011,2014,Vidhyalakshmi *et al.*, 2014**) for five variables and (**Gopalan *et al.*, 2012**) for six variables. In this paper the non-trivial integral solutions of the non-homogeneous equation with six unknowns given by $x^4 - y^4 - 8zw = 2T(P^2 + 1)$ are discussed. A few relations among the solutions are presented.

2. Method of Analysis

The Diophantine equation representing the given equation with six unknowns under consideration is

$$x^4 - y^4 - 8zw = 2T(P^2 + 1) \quad \text{.......... (1)}$$

The introduction of the linear transformations,

$$x = u + v, y = u - v, z = u^2, w = v^2, P = 2v, T = 4uv, \quad u, v > 0 \quad \text{.......... (2)}$$

in (1) leads to

$$u^2 - uv - (3v^2 + 1) = 0 \quad \text{.......... (3)}$$

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The above equation (3) is solved through different approaches and thus, one obtains different sets of solutions to (1)

2.1 Pattern1

Considering (3) as a quadratic in u and solving we get

$$u = \frac{v \pm \sqrt{13v^2 + 4}}{2}$$

By considering $v = 2X$ .... (4)

we get the solution of (3) as

$u = X \pm \sim$ .... (5)

where $\sim^2 = 13X^2 + 1$ .... (6)

The General solution $(\sim_n, X_n)$ of this pellian equation (6) is obtained as

$$\left.\begin{aligned} \sim_n &= \frac{1}{2}\left[\left(649+180\sqrt{13}\right)^{n+1} + \left(649-180\sqrt{13}\right)^{n+1}\right] \\ X_n &= \frac{1}{2\sqrt{13}}\left[\left(649+180\sqrt{13}\right)^{n+1} - \left(649-180\sqrt{13}\right)^{n+1}\right], n = 0,1,2.... \end{aligned}\right\} \quad \text{.. (7)}$$

Case1: Consider $u_n = X_n + \sim_n$ and $v_n = 2X_n$ .... (8)

Then the corresponding integral solutionS to (1) are given by

$$\left.\begin{aligned} x_n &= \frac{1}{2}f + \frac{3}{2\sqrt{13}}g \\ y_n &= \frac{1}{2}f - \frac{1}{2\sqrt{13}}g \\ z_n &= \frac{1}{52}(13f^2 + 2\sqrt{13}fg + g^2) \\ w_n &= \frac{1}{13}g^2 \\ T_n &= \frac{2}{13}(g^2 + \sqrt{13}fg) \\ P_n &= \frac{2}{\sqrt{13}}g \end{aligned}\right\} \quad \text{... (9)}$$

where

$$\left.\begin{aligned} f &= \left(649+180\sqrt{13}\right)^{n+1} + \left(649-180\sqrt{13}\right)^{n+1} \\ g &= \left(649+180\sqrt{13}\right)^{n+1} - \left(649-180\sqrt{13}\right)^{n+1} \end{aligned}\right\} \quad \text{... (10)}$$

A few numerical examples are given below

n	0	1	2
x_n	1189	1543321	2003229469
y_n	469	608761	790171309
z_n	687241	1157864233681	1950771976632751321
w_n	129600	218350598400	367877524885646400
T_n	1193760	2011249753920	3388557607903248480
P_n	720	934560	1213058160

The solutions (9) of (1) satisfy the following properties

- $52z_n - 20x_{2n+1} - 8y_{2n+1} - 24 = 0$
- $2(x_{2n+1} - y_{2n+1}) = (x_n + 3y_n)(x_n - y_n)$

- The following expressions are nasty numbers:

  (a) $21(52z_n - 13(x_{2n+1} - 3y_{2n+1}) + 4)$

  (b) $3(x_{2n+1} + 3y_{2n+1} + 4)$

- The following are cubic integers:

  (a) $18[26z_n - 10x_{2n+1} - 4y_{2n+1}]$

  (b) $14(x_{2n+1} + 3y_{2n+1} + 4) - 13(x_n - y_n)^2 - 104z_n + 26T_n$

  (d) $4[x_{3n+2} + 3y_{3n+2} + 3(x_n + 3y_n)]$

- The following expressions are biquadratic integers:

  (a) $8[x_{4n+3} + 3y_{4n+3} + 2(x_n + 3y_n)^2 - 4]$

  (b) $8[x_{4n+3} + 3y_{4n+3} + 4(x_{2n+1} + 3y_{2n+1}) + 12]$

Remarkable observations

I: By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the Parabola.

(a) Illustration1: It is to be noted that the Parabola

$$Y^2 = 2X$$

is satisfied for the following three sets of values of X and Y

Set1:

$Y = x_n + 3y_n$

$X = x_{2n+1} + 3y_{2n+1} + 4$

Set2:

$Y = x_{2n+1} + 3y_{2n+1} + 4$

$X = x_{4n+3} + 3y_{4n+3} + 2(x_n + 3y_n)^2 - 4$

Set3:

$X = w_n x_{2n+1} + 3y_{2n+1} w_n + 4w_n$

$Y = x_{2n+1} - y_{2n+1}$

(b) **Illustration2:** The Parabola

$$Y^2 = 4X$$

is satisfied for the following two sets of values of Y and X

Set1: $Y = (x_n - y_n)$

$X = w_n$

Set2: $Y = P_n$

$X = w_n$

Case2:

Consider $u_n = X_n - \sim_n$ and $v_n = 2X_n$ ………………………………………………………………….. (11)

Then the corresponding integral solutions to (1) are given by

$$\left.\begin{aligned} x_n &= \frac{3g}{2\sqrt{13}} - \frac{f}{2} \\ y_n &= -\left(\frac{g}{2\sqrt{13}} + \frac{f}{2}\right) \\ z_n &= \frac{1}{52}(13f^2 - 2\sqrt{13}fg + g^2) \\ w_n &= \frac{1}{13}g^2 \\ T_n &= \frac{2}{13}(g^2 - \sqrt{13}fg) \\ P_n &= \frac{2}{\sqrt{13}}g \end{aligned}\right\} \quad \ldots\ldots (12)$$

where f *and* g are obtained from (10)

A few numerical examples are given below:

n	0	1	2
x_n	-109	-141481	-183642229
y_n	-829	-1076041	-1396700389
z_n	219961	370589955121	6243706975667731481
w_n	129600	218350598400	367877524885646400
T_n	-675360	-1137847360320	1917047508360662880
P_n	720	934560	1213058160

2.2 Pattern2

Considering (3) as a quadratic in v and solving we get

$$v = \frac{-u \pm \sqrt{13u^2 - 12}}{6}$$

By considering $u = 2X$ ……………………………………………………………………………………… (13)

in the above equation we get

$$v = -\frac{1}{3}(X + \sim) \quad \ldots\ldots (14)$$

where $\sim^2 = 13X^2 - 3$ ……………………………………………………………………………………… (15)

The smallest positive integer solution $\left(X_0, \sim_0\right)$ of (15) is

$$X_0 = 2, \sim_0 = 7 \quad \ldots\ldots (16)$$

To obtain the other solutions of (15), consider the Pellian equation

$$\sim^2 = 13X^2 + 1 \quad \ldots\ldots (17)$$

whose general solution $(\tilde{X}_n, \tilde{\sim}_n)$ is given by

$$\begin{aligned} \tilde{\sim}_n &= \frac{1}{2}\left[(649+180\sqrt{13})+(649-180\sqrt{13})^{n+1}\right] \\ \tilde{X}_n &= \frac{1}{2}\left[(649+180\sqrt{13})+(649-180\sqrt{13})^{n+1}\right], n = 1,2,3..... \end{aligned} \quad \ldots\ldots (18)$$

Applying Brahmagupta lemma, between the solutions $\left(X_0, \sim_0\right)$ and, $(\tilde{X}_n, \tilde{\sim}_n)$ the general solutions of (15) are found to be

$$\left.\begin{aligned} X_{n+1} &= 2\tilde{\sim}_n + 7\tilde{X}_n \\ \sim_{n+1} &= 7\tilde{\sim}_n + 26\tilde{X}_n \end{aligned}\right\}$$

Using (16) and (18), the above equations becomes

$$\left.\begin{aligned} X_{n+1} &== f + \frac{7g\sqrt{13}}{26} \\ \sim_{n+1} &= \frac{7f}{2} + g\sqrt{13} \end{aligned}\right\} \quad \ldots\ldots (19)$$

Taking advantage of (19), (13), (14) & (2), and performing some algebra, the sequence of integral solutions of (1) can be obtained as

$$\left.\begin{aligned} x_{n+1} &= \frac{1}{2}f + \frac{3}{26}g\sqrt{13} \\ y_{n+1} &= \frac{7}{2}f + \frac{25}{26}g\sqrt{13} \\ z_{n+1} &= 4[f^2 + \frac{7}{13}\sqrt{13}fg + \frac{49}{52}g^2] \\ w_{n+1} &= \frac{1}{9}[\frac{81f^2}{4} + \frac{1089}{52}g^2 + \frac{297}{26}\sqrt{13}fg] \\ T_{n+1} &= -\frac{8}{3}[\frac{9f^2}{2} + \frac{231}{52}g^2 + \frac{129}{52}\sqrt{13}gf] \\ P_{n+1} &= -\frac{2}{3}[\frac{9f}{2} + \frac{33}{26}\sqrt{13}g] \end{aligned}\right\} \quad \ldots\ldots (20)$$

where f *and* g are obtained from (10)

A few numerical examples are given below:

n	0	1	2
x_n	1	1189	1543321
y_n	7	9043	11737807
z_n	16	26173456	44097090238096
w_n	9	15421329	25981886201049
T_n	-48	-80362128	-135394273460208
P_n	-6	-7854	-10194486

The patterns of solutions $(x_n, y_n, z_n, w_n, P_n, T_n)$ to (1) presented in (9), (12) and (20) satisfy the following recurrence relations:

$$\left.\begin{aligned} x_{n+2}-1298x_{n+1}+x_n &= 0 \\ y_{n+2}-1298y_{n+1}+y_n &= 0 \\ z_{n+2}-1684802z_{n+1}+z_n &= -777600 \\ w_{n+2}-1684802w_{n+1}+w_n &= 259200 \\ P_{n+2}-1298P_{n+1}+P_n &= 0 \\ T_{n+2}-1684802T_{n+1}+T_n &= 518400 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

3. Conclusion

In this paper, different patterns of non-zero distinct integral solutions of the non-omogeneous biquadratic equation with six unknowns, given in the title are obtained. As the biquadratic equations are rich in variety, one may attempt for finding integer solutions to the biquadratic equations with multiple variables and search for their properties.

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