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STABILITY ANALYSIS FROM FOUTH ORDER EVOLUTION EQUATIONS FOR TWO GRAVITY WAVE PACKETS IN THE PRESENCE OF WIND FLOWING OVER WATER

***Dhar, A. K. and Mondal, J.**

Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, P.O. Botanic Garden, Howrah - 711103, West Bengal, India

***Corresponding Author**

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Abstract

Starting from Zakharov integral equation, two coupled fourth order nonlinear evolution equations have been derived in deep water for two gravity wave packets propagating in the same direction in the presence of wind blowing over water. On the basis of these evolution equations, the stability analysis is made for a uniform gravity wave train in the presence of another wave train having the same group velocity. Graphs are plotted for growth rate of instability against the perturbation wave number for different values of the amplitudes of two wave trains and for different values of wind velocity.

Keywords: Nonlinear Evolution Equations, Gravity Wave

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INTRODUCTION

One approach to studying the stability of finite amplitude surface gravity waves in deep water is through the application of the lowest order nonlinear evolution equation, which is the nonlinear Schrodinger equation. Zakharov's (1968) study is along this line, allowing for finite amplitude wave trains to be subjected to modulational perturbations in two horizontal directions both along and perpendicular to the direction of the wave train. Davey and Stewartson (Davey and Stewartson, 1974) made an extension of this to water of finite depth. Further extensions of this were made by Djordjevic and Redekopp (1977) to include capillarity and by Das (1986) to include density stratification.

For small amplitude, $ka<0.1$ the predictions from the nonlinear Schrodinger equation when compared with Longuet-Higgin's (1978) exact results are fairly accurate.

But for $ka>0.15$ the predictions from the nonlinear Schrodinger equation do not agree with the exact results of Longuet-Higgins (1978). Dysthe (1979) has shown that a stability analysis made from a fourth-order nonlinear evolution equation that is one order higher than the nonlinear Schrodinger equation gives results consistent with the exact results of Longuet-Higgin's (1978) and with the experimental results of Benjamin--Feir (1967) for wave steepness up to 0.25. From this fourth-order evolution equation Janssen (1983) has elaborated on the Dysthe (1979) approach by investigating the effect of wave-induced flow on the long time behavior of Benjamin-Feir (1967) instability and has also applied this equation to the homogeneous random field of gravity waves and obtained the nonlinear energy transfer function found by Dungey and Hui (1979) Stiassnie (1984) has shown that Zakharov's (1968) integral equation yields the modified or fourth order nonlinear Schrodinger equation for the particular case of narrow spectrum.

Hogan (1985) has considered the stability of a train of nonlinear capillary-gravity waves on the surface of an ideal fluid of infinite depth. He derived from the Zakharov's (1968) equation under the assumption of a narrow band of waves and including the full from of interaction coefficient for capillary-gravity waves, an evolution equation for the wave envelope that is correct to fourth order in the wave steepness. Fourth order nonlinear evolution equation for deep water surface-gravity waves in different contexts and stability analysis made from them were derived by Dhar and Das (1990, 1994), Debsarma and Das (2002), Hara and Mei (1991, 1994), Bhattacharyya and Das (1997). What has been said in the previous paragraphs is for the evolution equation of a single wave packet. It is of considerable importance to extend the stability analysis of a wave packet in the presence of another wave packet. Stability analysis of a surface gravity wave in deep water in the presence of a second wave has been made by Roskes (1976) based on the lowest-order nonlinear Schrodinger equations. In his investigation modulational perturbation is restricted to a direction along which group velocity projections of the two waves overlap and it is argued that the modulation will grow at a faster rate along this direction when $0< \theta < 70.5^{\circ}$, where θ is the angle between the two propagation directions of two waves.

Dhar and Das (1991) made the same analysis of Roskes (1976) making use of two coupled fourth-order nonlinear evolution equations that they derived for two wave packets having the same characteristic wave number. The same analysis including the effect of capillarity was later made by Dhar and Das (1993) using the multiple scale method. They observed significant deviations from the results obtained from coupled cubic nonlinear Schrodinger equations. Pierce and Knobloch (1994) derived third order evolution equations for counter propagating capillary- gravity wave trains having equal characteristic wave number and frequency propagating over finite depth water. The resulting equations are asymptotically exact and nonlocal. In the present paper two coupled fourth order nonlinear evolution equations are derived in deep water for two gravity wave packets propagating in the same direction with unequal wave numbers in the presence of wind flowing over water. Here we have used a general method, based on Zakharov integral equation.

Unlike Dhar and Das (1991, 1993), the evolution equations are derived here using Zakharov integral equation. Stiassnie (1984) and Hogan (1988) also used the Zakharov integral equation for the derivation of fourth order nonlinear evolution equations for a surface gravity wave packet and capillary-gravity wave packet respectively. In deriving the two coupled evolution equations, we make an extension of the paper by Dhar and Das (1991) who derived the fourth order nonlinear evolution equations for two gravity wave packets with equal wave numbers using multiple scale method. The expression for the change in phase speed for the case of gravity waves was first obtained by Longuet-Higgins and Phillips (1962) by the perturbation method. Onorato *et al*. (2006) also derived third-order evolution equations to study the problem of interaction of two wave systems in deep water with equal characteristic wave number and propagating in two different directions. They found that the introduction of a second wave results in an increase of the instability growth rates and causes enlargement of the instability region.

In our paper the relative changes in phase speed of each uniform wave train in the presence of another one have been derived. On the basis of two coupled nonlinear Schrodinger equations, the stability analysis is made of a uniform gravity wave train in the presence of another uniform gravity wave train, when the group velocities of the two wave trains coincide. The instability condition and an expression for the growth rate of instability are then obtained. Stable-unstable regions of the second wave train have been plotted for different values of the first wave train and for different values of wind velocity. We have also plotted the growth rate of instability against perturbation wave number for different values of the amplitudes of two wave trains and for different values of wind velocity.

Basic equations

We take the common horizontal interface between water and air in the undisturbed state as z=0 plane. In the undisturbed state air flows over water with a velocity u in a direction that is taken as the x- axis. The equation of the common interface is taken as $z=\zeta(x,t)$ at any time t in the perturbed state.

The perturbed velocity potentials $\phi=(x,z,t)$ and $\phi'=(x,z,t)$ of water and air respectively satisfy the following Laplace equations

$$\nabla^2\phi=0 \quad \text{in} \quad -\infty<z<\zeta \qquad (1)$$

$$\nabla^2\phi'=0 \quad \text{in} \quad \zeta<z<\infty \qquad (2)$$

The kinematic boundary conditions to be satisfied at the interface are the following

$$\frac{\partial\phi}{\partial z}-\frac{\partial\zeta}{\partial t}=\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} \quad \text{at} \quad z=\zeta \qquad (3)$$

$$\frac{\partial\phi'}{\partial z}-\frac{\partial\zeta}{\partial t}-u\frac{\partial\zeta}{\partial x}=\frac{\partial\phi'}{\partial x}\frac{\partial\zeta}{\partial x} \quad \text{at} \quad z=\zeta \qquad (4)$$

The condition of continuity of pressure at the interface is given by

$$\frac{\partial\phi}{\partial t}-\gamma\frac{\partial\phi'}{\partial t}+(1-\gamma)g\zeta-\gamma u\frac{\partial\phi'}{\partial x}=\frac{1}{2}(\Delta\phi)^2+\frac{\gamma}{2}(\Delta\phi')^2 \quad \text{at} \quad z=\zeta \qquad (5)$$

Where $\gamma = \frac{\rho}{\rho'}$ is the ratio of densities of air to water and g is the acceleration due to gravity.

Finally the velocity potentials ϕ and ϕ' should satisfy the following conditions at infinity

$$\phi \to 0 \quad \text{as} \quad z \to -\infty \qquad (6)$$

$$\phi' \to 0 \quad \text{as} \quad z \to \infty \qquad (7)$$

As the disturbance is assumed to be a progressive wave we look for solutions of the above equations in the following form

$$G = G_{00} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[G_{mn} \exp i(m\psi_1 + n\psi_2) + G^*_{mn} \exp - i(m\psi_1 + n\psi_2)] \qquad (8)$$

where G stands for $\phi, \phi'\ \zeta$ and $\psi_1 = k_1 x - \omega\, t,\quad \psi_1 = k_1 x - \omega\, t$. In the above summation on the right hand side of equation (8), $(m,n) \neq (0,0)$. The Fourier coefficients $\phi_{00,}\ \phi'_{00,}\ \phi_{mn,}\ \phi'_{mn,}\ \phi^*_{mn}\ \phi'^*_{mn,}$ are functions of z, $x_1 = \varepsilon x, t_1 = \varepsilon\, t$ and $\zeta_{00,}\ \zeta_{mn,}\ \zeta^*_{mn,}$ are functions of x_1, t_1 .Here * denotes complex conjugate, ε is a small ordering parameter measuring the weakness of wave steepness and ω, k satisfy the following linear dispersion relation for gravity waves

$$(1+\gamma)\omega^2 - 2\gamma\,\omega\, k\, u + \gamma k^2 u^2 - (1-\gamma) g k = 0 \qquad (9)$$

Derivation of evolution equations

The two coupled nonlinear evolution equations are derived here using Zakharov's integral equation which is given by

$$i\frac{\partial P(k,t)}{\partial t} = \iiint_{-\infty}^{\infty} (k,k_1,k_2,k_3) P^*(k_1,t) P(k_2,t) P(k_3,t)$$
$$\times \delta(k + k_1 - k_2 - k_3) \exp[i\{\omega(k) + \omega(k_1) - \omega(k_2) - \omega(k_3)\}t] dk_1\, dk_2\, dk_3 \qquad (10)$$

where $P(k,t)$ is related is related to the to the free surface elevation $\zeta(x,t)$ by

$$\zeta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{\frac{|k|}{2\omega(k)}\right\}^{\frac{1}{2}} \{P(k,t)\exp[i\{k.x - \omega t\}] + c.c\}\, dk \qquad (11)$$

In the above $x = (x, y)$ is the horizontal spatial vector $k = (k, l)$ is the wave vector, c.c. denotes complex conjugate and the kernel $T(k, k_1, k_2, k_3)$ is a scalar function used by Krasitskii (18).

The linearized wave frequency $\omega(k)$ connected to k through the following linear dispersion relation given by

$$\omega(k) = \left[\frac{|k|^{\frac{1}{2}}}{1+\gamma}\left\{\gamma\,|k|^{\frac{1}{2}}\, u + [(1-\gamma^2)g - \gamma\,|k|\,u^2]^{\frac{1}{2}}\right\}\right] \qquad (12)$$

A non zero contribution to the integral equation (10) can be obtained, when the following condition for four wave numbers is satisfied.

$$k + k_1 - k_2 - k_3 = 0 \quad (13)$$

Now we consider two narrow gravity wave packets centered around the wave vectors k_a and called the first and k_b second wave packet respectively. With $k = k_a$ the condition (13) is satisfied for two waves with wave vectors k_a and k_b in three cases given by $(a)\, k_1 = k_b, \quad k_2 = k_b, \quad k_3 = k_a, \quad (b)\, k_1 = k_3 = k_b, \quad k_2 = k_a, \quad (c)\, k_1 = k_2 = k_3 = k_a$

For obtaining the evolution equation of the first wave packet we take $k = k_a + e$ in equation (10) and introducing new variables $Q_1(e,t)$ and $Q_2(e,t)$ defined by

$$Q_1(e,t) = P(k_a + e, t)\exp[-i\{\omega(k_a + e) - \omega(k_a)\}t]$$
$$Q_2(e,t) = P(k_b + e, t)\exp[-i\{\omega(k_b + e) - \omega(k_b)\}t] \quad (14)$$

Equation (10) can be written as

$$\begin{aligned} & i\frac{\partial Q_1(e,t)}{\partial t} - Q_1(e,t)[\omega(k_a + e) - \omega(k_a)] \\ & = \iiint_{-\infty}^{\infty} T(k_a + e, k_b + e_1, k_b + e_2, k_a + e_3) Q^*{}_2(e_1,t) Q_2(e_2,t) Q_1(e_3,t) \\ & \times \delta(e + e_1 - e_2 - e_3) de_1\, de_2\, de_3 \\ & + \iiint_{-\infty}^{\infty} T(k_a + e, k_b + e_1, k_a + e_2, k_b + e_3) Q^*{}_1(e_1,t) Q_1(e_2,t) Q_2(e_3,t) \\ & \times \delta(e + e_1 - e_2 - e_3) de_1\, de_2\, de_3 \\ & + \iiint_{-\infty}^{\infty} T(k_a + e, k_a + e_1, k_a + e_2, k_a + e_3) Q^*{}_1(e_1,t) Q_1(e_2,t) Q_1(e_3,t) \\ & \times \delta(e + e_1 - e_2 - e_3) de_1\, de_2\, de_3 \end{aligned} \quad (15)$$

in which we replace $k_1 = k_b + e_1, \quad k_2 = k_b + e_2, \quad k_3 = k_a + e_3$ for the first triple integral $k_1 = k_b + e_1, \quad k_2 = k_a + e_2, \quad k_3 = k_b + e_3$ for the second and $k_1 = k_a + e_1, \quad k_2 = k_a + e_2, \quad k_3 = k_a + e_3$ finally for the third.

The surface elevations $\alpha_1(x,t)$ and $\alpha_2(x,t)$ for the first and second wave packets respectively for the new variables becomes

$$\begin{aligned} \alpha_1(x,t) = & \sqrt{\frac{1+\gamma}{2}}\frac{1}{2\pi}\exp i[k_a.x - \omega(k_a)t].\int_{-\infty}^{\infty} Q_1(e,t)\exp i(e.x) \\ & \times \left[\frac{|k_a + e|^{\frac{1}{4}}}{[\gamma u\,|k_a + e|^{\frac{1}{2}} + \{(1-\gamma^2)g - \gamma u^2\,|k_a + e|^{\frac{1}{2}}\}]^{\frac{1}{2}}}\right] de + c.c \\ & = \eta_1(x,t)\exp[i\{k_1.x - \omega(k_1)t\}] + c.c \end{aligned} \quad (16)$$

$$\alpha_2(x,t)=\sqrt{\frac{1+\gamma}{2}}\frac{1}{2\pi}\exp i[k_b.x-\omega(k_b)t].\int_{-\infty}^{\infty}Q_2(e,t)\exp i(e.x)$$

$$\times\left[\frac{|k_b+e|^{\frac{1}{4}}}{[\gamma u|k_b+e|^{\frac{1}{2}}+\{(1-\gamma^2)g-\gamma u^2|k_b+e|^{\frac{1}{2}}\}]^{\frac{1}{2}}}\right]de+c.c \quad \text{.....} \quad (17)$$

$$=\eta_2(x,t)\exp[i\{k_2.x-\omega(k_2)t\}]+c.c$$

where

$$\eta_1(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mu_1(e,t)\exp(ie.x)de$$

$$\eta_2(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mu_2(e,t)\exp(ie.x)de \quad \text{.....} \quad (18)$$

$$\mu_1(e,t)=\sqrt{\frac{1+\gamma}{2}}\frac{k_1^{\frac{1}{4}}}{d_1^{\frac{1}{4}}}(1+d_1'e)Q_1(e,t)$$

and

$$\mu_2(e,t)=\sqrt{\frac{1+\gamma}{2}}\frac{k_2^{\frac{1}{4}}}{d_2^{\frac{1}{4}}}(1+d_2'e)Q_2(e,t) \quad \text{.....} \quad (19)$$

$$d_1=[\{(1-\gamma^2)g-\gamma u^2k_1\}^{\frac{1}{2}}+\gamma uk_1^{\frac{1}{2}}]$$

$$d_1'=\frac{1}{4k_1}-\frac{1}{4}[\frac{\gamma u}{\sqrt{k_1}}-\frac{\gamma u^2}{\{(1-\gamma^2)g-\gamma u^2k_1}]$$

$$\times\frac{1}{\{(1-\gamma^2)g-\gamma u^2k_1\}^{\frac{1}{2}}+\gamma uk_1^{\frac{1}{2}}\}}$$

$$d_2=[\{(1-\gamma^2)g-\gamma u^2k_2\}^{\frac{1}{2}}+\gamma uk_2^{\frac{1}{2}}]$$

$$d_2'=\frac{1}{4k_2}-\frac{1}{4}[\frac{\gamma u}{\sqrt{k_2}}-\frac{\gamma u^2}{\{(1-\gamma^2)g-\gamma u^2k_2}]$$

$$\times\frac{1}{\{(1-\gamma^2)g-\gamma u^2k_2\}^{\frac{1}{2}}+\gamma uk_2^{\frac{1}{2}}\}}$$

Using equations (16) ,(18) and (19) we have first evaluated the left hand side of equation (15) by setting $k_1+e=(k_1'+e)\overline{x}$ in $\omega(k_1+e)$ and expanding in powers of e up to third degree, we get the following expression of left hand side of equation (15) after evaluation of Fourier inversion integrals. We also set $e_i=e_1'\overline{x}$ $(i=1,2,3)$ in the arguments of T appearing on the right hand side of equation (15). Now we make Taylor expansions of them in powers of e_1' $(i=1,2,3)$ up to first degree in these variables in which we have used the following notations

$$\omega_i=\omega(k_i),\quad(i=1,2)\quad m=\frac{k_2}{k_1},\quad n=\frac{\omega_2}{\omega_1},\quad c_g=\frac{d\omega(k)}{dk}$$

Now we introduce the following dimensionless variables

$\eta_1' = k_1\eta_1, \quad \eta_2' = k_2\eta_2, \quad x' = k_2 x, \quad t' = \omega_2 t, \quad v' = \sqrt{\frac{k_2}{g}}\,u$ in the expressions of left hand and right side of the reduced form of equation (15). Now deliting the primes and taking the Fourier inversion integral of expression of right hand side of the reduced form of equation (15) we get the following nonlinear evolution equation for the first wave packet in the presence of second wave packet.

$$
\begin{aligned}
& i\left(\frac{\partial\eta_1}{\partial t} + \delta_1^{(1)}\frac{\partial\eta_1}{\partial x}\right) + \delta_2^{(1)}\frac{\partial^2\eta_1}{\partial x^2} + i\delta_3^{(1)}\frac{\partial^3\eta_1}{\partial x^3} = \gamma_1^{(1)}\eta_1{}^2\eta^*{}_1 + i\gamma_2^{(1)}\eta_1\eta^*{}_1\frac{\partial\eta_1}{\partial x} \\
& + i\gamma_3^{(1)}\eta_1{}^2\frac{\partial\eta_1^*}{\partial x} + \gamma_4^{(1)}\eta_1 H\left[\frac{\partial}{\partial x}(\eta_1\eta_1^*)\right] + \beta_1^{(1)}\eta_1\eta_2\eta_2^* + i\beta_2^{(1)}\eta_2\eta_2^*\frac{\partial\eta_1}{\partial x} \\
& + i\beta_3^{(1)}\eta_1\eta_2^*\frac{\partial\eta_2}{\partial x} + i\beta_4^{(1)}\eta_2\eta_2\frac{\partial\eta_2^*}{\partial x} + \beta_5^{(1)}\eta_1 H\left[\frac{\partial}{\partial x}(\eta_2\eta_2^*)\right]
\end{aligned} \qquad (20)
$$

where H is the Hilbert transform operator given by

$$H(\psi) = \frac{1}{\pi} P\int_{-\infty}^{\infty}\frac{\psi(\xi)}{\xi - x}d\xi \qquad (21)$$

The coefficients $\delta_i^{(1)} \quad (i = 1,2,3)$, $\gamma_i^{(1)} \quad (i = 1,2,3,4)$ and $\beta_i^{(1)} \quad (i = 1,2,3,4,5)$ appearing in equation (20) are given in the Appendix. In the absence of the second wave and for $v = \gamma = 0$ the coupled equations reduce to equation (14) of Stiassnie (25) for a single gravity wave train. Also in the absence of second wave, the equation (20) reduces to equation (34) of Dhar and Das (1990) for a single gravity wave train. Proceeding in the same way and making an interchange between the suffixes a' and b' in the evolution equation (15), we obtain the following nonlinear evolution equation for the second wave packet in the presence of first wave packet.

$$
\begin{aligned}
& i\left(\frac{\partial\eta_2}{\partial t} + \delta_1^{(2)}\frac{\partial\eta_2}{\partial x}\right) + \delta_2^{(2)}\frac{\partial^2\eta_2}{\partial x^2} + i\delta_3^{(2)}\frac{\partial^3\eta_2}{\partial x^3} = \gamma_1^{(2)}\eta_2{}^2\eta^*{}_2 + i\gamma_2^{(2)}\eta_2\eta^*{}_2\frac{\partial\eta_2}{\partial x} \\
& + i\gamma_3^{(2)}\eta_2{}^2\frac{\partial\eta_2^*}{\partial x} + \gamma_4^{(2)}\eta_2 H\left[\frac{\partial}{\partial x}(\eta_2\eta_2^*)\right] + \beta_1^{(2)}\eta_2\eta_1\eta_1^* + i\beta_2^{(2)}\eta_1\eta_1^*\frac{\partial\eta_2}{\partial x} \\
& + i\beta_3^{(2)}\eta_2\eta_1^*\frac{\partial\eta_1}{\partial x} + i\beta_4^{(2)}\eta_2\eta_1\frac{\partial\eta_1^*}{\partial x} + \beta_5^{(2)}\eta_2 H\left[\frac{\partial}{\partial x}(\eta_1\eta_1^*)\right]
\end{aligned} \qquad (22)
$$

where the coefficients $\delta_i^{(2)} \quad (i = 1,2,3)$, $\gamma_i^{(2)} \quad (i = 1,2,3,4)$ and $\beta_i^{(2)} \quad (i = 1,2,3,4,5)$ are given in the Appendix.

Stability of finite amplitude wave trains

The coupled equations (20) and (22) admit the following uniform wave train solutions

$$\eta_1 = \eta_{01}\exp(-\Delta\omega_1 t) \qquad (23)$$

$$\eta_2 = \eta_{02}\exp(-\Delta\omega_2 t) \qquad (24)$$

where η_{01} and η_{02} are real constants. Substituting equations (23) and (24) in equations (20) and (22) respectively, the amplitude dependent nonlinear frequency shifts of the two waves $\Delta\omega_1$ and $\Delta\omega_2$ are given by

$$
\begin{aligned}
\Delta\omega_1 &= \gamma_1^{(1)}\eta_{01}{}^2 + \beta_1^{(1)}\eta_{02} \\
\Delta\omega_2 &= \gamma_1^{(2)}\eta_{02}{}^2 + \beta_1^{(2)}\eta_{01}
\end{aligned} \qquad (25)
$$

The dimensionless wave numbers of the first and second wave are k_1/k_2 and 1 respectively. Therefore the amplitude dependent shifts in phase speeds Δc_1 and Δc_2 of the two waves are the following

$$\Delta c_1 = \frac{\Delta\omega_1}{k_1/k_2} = m\left(\gamma_1^{(1)}\eta_{01}^{\ 2} + \beta_1^{(1)}\eta_{02}\right)$$
$$\Delta c_2 = \Delta\omega_2 = m\left(\gamma_1^{(2)}\eta_{02}^{\ 2} + \beta_1^{(2)}\eta_{01}\right) \quad \ldots (26)$$

To make a stability analysis we consider perturbations of the following form

$$\eta_1 = \eta_{01}\left[1 + \eta_1'(\xi,t)\right]\exp(-\Delta\omega_1 t) \quad \ldots (27)$$

$$\eta_2 = \eta_{02}\left[1 + \eta_2'(\xi,t)\right]\exp(-\Delta\omega_2 t) \quad \ldots (28)$$

where $\eta_1'(\xi,t)$, $\eta_2'(\xi,t)$ are small perturbations of amplitudes η_1 and η_2 respectively. Substituting equations (27) and (28) in two evolution equations (20) and (22) respectively and then linearizing with respect to η_1' and η_2' we get the following two equations

$$i\left(\frac{\partial\eta_1'}{\partial t} + \delta_1^{(1)}\frac{\partial\eta_1'}{\partial x}\right) + \delta_2^{(1)}\frac{\partial^2\eta_1'}{\partial x^2} + i\delta_3^{(1)}\frac{\partial^3\eta_1'}{\partial x^3} = \gamma_1^{(1)}\eta_1'^2\eta_1'^* + i\gamma_2^{(1)}\eta_1'\eta_1'^*\frac{\partial\eta_1'}{\partial x}$$
$$+ i\gamma_3^{(1)}\eta_1'^2\frac{\partial\eta_1'^*}{\partial x} + \gamma_4^{(1)}\eta_1' H\left[\frac{\partial}{\partial x}(\eta_1'\eta_1'^*)\right] + \beta_1^{(1)}\eta_1'\eta_2'\eta_2'^* + i\beta_2^{(1)}\eta_2'\eta_2'^*\frac{\partial\eta_1'}{\partial x} \quad \ldots (29)$$
$$+ i\beta_3^{(1)}\eta_1'\eta_2'^*\frac{\partial\eta_2'}{\partial x} + i\beta_4^{(1)}\eta_2'\eta_2'\frac{\partial\eta_2'^*}{\partial x} + \beta_5^{(1)}\eta_1' H\left[\frac{\partial}{\partial x}(\eta_2'\eta_2'^*)\right]$$

$$i\left(\frac{\partial\eta_2'}{\partial t} + \delta_1^{(2)}\frac{\partial\eta_2'}{\partial x}\right) + \delta_2^{(2)}\frac{\partial^2\eta_2'}{\partial x^2} + i\delta_3^{(2)}\frac{\partial^3\eta_2'}{\partial x^3} = \gamma_1^{(2)}\eta_2'^2\eta_2'^* + i\gamma_2^{(2)}\eta_2'\eta_2'^*\frac{\partial\eta_2'}{\partial x}$$
$$+ i\gamma_3^{(2)}\eta_2'^2\frac{\partial\eta_2'^*}{\partial x} + \gamma_4^{(2)}\eta_2' H\left[\frac{\partial}{\partial x}(\eta_2'\eta_2'^*)\right] + \beta_1^{(2)}\eta_2'\eta_1'\eta_1'^* + i\beta_2^{(2)}\eta_1'\eta_1'^*\frac{\partial\eta_2'}{\partial x} \quad \ldots (30)$$
$$+ i\beta_3^{(2)}\eta_2'\eta_1'^*\frac{\partial\eta_1'}{\partial x} + i\beta_4^{(2)}\eta_2'\eta_1'\frac{\partial\eta_1'^*}{\partial x} + \beta_5^{(2)}\eta_2' H\left[\frac{\partial}{\partial x}(\eta_1'\eta_1'^*)\right]$$

Now setting $\eta_1' = \eta_r^{(1)} + i\eta_i^{(1)}$ and $\eta_2' = \eta_r^{(2)} + \eta_r^{(2)}$ in the above two equations (29) and (30) respectively where $\eta_r^{(1)}, \eta_i^{(2)}, \eta_r^{(1)}, \eta_i^{(2)}$ are real and then assuming the space time dependence of $\eta_r^{(1)}, \eta_r^{(2)}, \eta_i^{(1)}, \eta_i^{(2)}$ is of the form $\exp i(\mu x - \Omega t)$ and finally equating real and imaginary parts on both sides of each equation we get the four coupled equations. Neglecting higher order terms, the condition for the existence of a nontrivial solution to the above four algebraic equations gives the following nonlinear dispersion relation

$$\left[(\Omega - C_1)^2 - P_1E_1\right]\left[(\Omega - C_2)^2 - P_2E_2\right] = H_1(\Omega - C_1)(\Omega - C_2) - F_1(\Omega - C_1) - F_2(\Omega - C_2) \quad \ldots (31)$$

where

$$C_1 = \delta_1^{(1)}\mu - \delta_3^{(1)}\mu^3 - \gamma_2^{(1)}\mu\eta_{01}^{\ 2} - \beta_2^{(1)}\mu\eta_{02}^{\ 2}, \quad C_2 = \delta_1^{(2)}\mu - \gamma_2^{(2)}\mu\eta_{02}^{\ 2} - \beta_2^{(2)}\mu\eta_{01}^{\ 2}$$
$$F_1 = 2\delta_2^{(2)}\left\{\beta_1^{(2)}(\beta_1^{(2)} + \beta_4^{(1)}) + \beta_1^{(1)}(\beta_3^{(2)} - \beta_4^{(2)})\right\}\mu^3\eta_{01}^{\ 2}\eta_{02}^{\ 2}$$
$$F_2 = 2\delta_2^{(2)}\left\{\beta_1^{(1)}(\beta_1^{(1)} + \beta_4^{(1)}) + \beta_1^{(2)}(\beta_3^{(1)} - \beta_4^{(1)})\right\}\mu^3\eta_{01}^{\ 2}\eta_{02}^{\ 2} \qquad \text{.........} \quad (32)$$
$$H_1 = 2\left(\beta_3^{(1)}\beta_3^{(2)} + \beta_4^{(1)}\beta_4^{(2)}\right)\mu^3\eta_{01}^{\ 2}\eta_{02}^{\ 2}$$
$$H_2 = 4\delta_2^{(1)}\delta_2^{(2)}\left\{\beta_1^{(1)}\beta_1^{(2)} - (\beta_1^{(1)}\beta_5^{(2)} + \beta_1^{(1)}\beta_5^{(2)})\mu\right\}\mu^3\eta_{01}^{\ 2}\eta_{02}^{\ 2}$$

If we assume nearly equal group velocity of the two waves i.e., we assume $\delta_1^{(1)} \approx \delta_1^{(2)}$ then it can be shown from two evolution equations that $\Omega - \delta_1^{(1)}\mu = O(\varepsilon^2)$ and $\Omega - \delta_1^{(2)}\mu = O(\varepsilon^2)$, where ε is a small ordering parameter, the smallness of η_{01}, η_{02} and μ. The nonlinear dispersion relation (31) at fourth order can be solved for the second wave train in the presence of the first wave train as follows:

$$\left[(\Omega - C_1) + 0.5F_2 / \{(\Omega^{(2)} - C_2)^2 - P_1E_1\}\right]^2 = P_2E_2 + \{H_2 - F_1(\Omega^{(2)} - \delta_1^{(1)}\mu)\} / \{(\Omega^{(2)} - C_2)^2 - P_1E_1\} \qquad \text{.........} \quad (33)$$

where $\Omega^{(1)}$ and $\Omega^{(2)}$ are the solutions of the dispersion relation (31) for the first and second wave trains at the lowest order given by

$$\Omega^{(j)} = \delta_1^{(j)}\mu \pm \left\{\delta_2^{(j)}\mu^2(\delta_2^{(j)}\mu^2 + 2\gamma_1^{(j)})\eta_{0j}^{\ 2}\right\}^{\frac{1}{2}}, \qquad (j = 1,2) \qquad \text{.........} \quad (34)$$

The instability condition from equation (33) of the second wave train in the presence of first wave train is given by

$$P_2E_2 + \{H_2 - F_1(\Omega^{(2)} - \delta_1^{(1)}\mu)\} / \{(\Omega^{(2)} - C_2)^2 - P_1E_1\} < 0 \qquad \text{.........} \quad (35)$$

The above instability condition in the absence of first wave train becomes

$$P_2E_2 < 0$$

that is, $\delta_2^{(2)}\mu^2\left\{\delta_2^{(2)}\mu^2 + 2(\gamma_1^{(2)} - \gamma_1^{(2)}\,|\,\mu\,|)\eta_{02}^{\ 2}\right\} < 0$ .... (36)

which is similar to the instability condition of single wave packet. The instability condition (36) is identical with the instability condition (57) of Dhar and Das (1990). Also for $\nu = \gamma = 0$ the above instability condition reduces to equation (3.8) of Dysthe (1979).

Stable-unstable regions of the second wave train in the presence of first wave train are shown in figure 1 for different values of wind velocity. We have also plotted the marginal stability curves of the second wave train in the absence of the first wave train in figure 2 for different values of wind velocity ν.

It is found that the instability region of the second wave train expands due to the presence of the first wave train for fixed value of wind velocity. We also observe that the instability region is shortened slightly by the inclusion of fourth order terms. Further with the increase of wind velocity, the instability region is again shortened for fixed value of the amplitude of the first wave train. The growth rate of instability I_G of the second wave train of longer wavelength is given by

$$I_G = \left[-P_2E_2 - \frac{H_2 - F_1(\Omega^{(2)} - \delta_1^{(1)}\mu)}{(\Omega^{(2)} - C_2)^2 - P_1E_1}\right]^{\frac{1}{2}} \qquad \text{.........} \quad (37)$$

We have plotted in figures 3 and 4 the growth rate of instability I_G of the second wave train against the perturbation wave number in the presence of first wave train and for different values of wind velocity. We have plotted in figure 5 the similar curves in the absence of first wave train and also have plotted the corresponding curves that can be obtained from third order evolution equations. From these figures, it is found that the growth rate of instability of the second wave train increases due to the presence of the first wave train and it increases with the increase of the amplitude of the first wave train for fixed value of wind velocity. We also observe that the influence of fourth order terms is to increase the growth rate of instability. Further the growth rate of instability increases with the increase of wind velocity for fixed value of the amplitude of the first wave train.

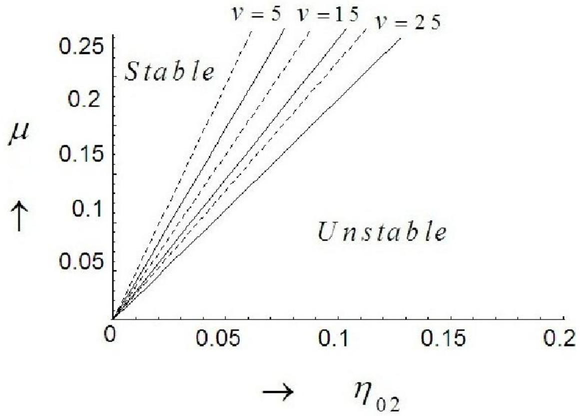

Fig. 1. Stable-unstable regions of the second wave train for some different values of dimensionless wind velocity v written on the graphs. Here $\eta_{01} = 0.2, \quad (k_1, k_2) = (0.7132, 0.3394)$.----------represents fourth order results and .... represents third order results

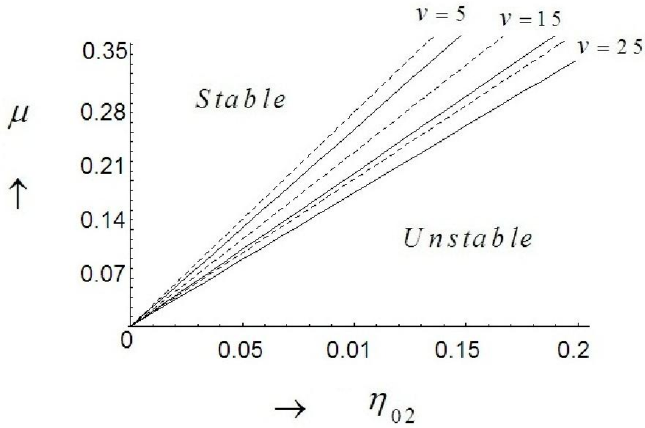

Fig. 2. Stable-unstable regions of the second wave train for some different values of dimensionless wind velocity v written on the graphs. Here $\eta_{01} = 0, \quad (k_1, k_2) = (0.7132, 0.3394)$.----------- represents fourth order results and .... represents third order results

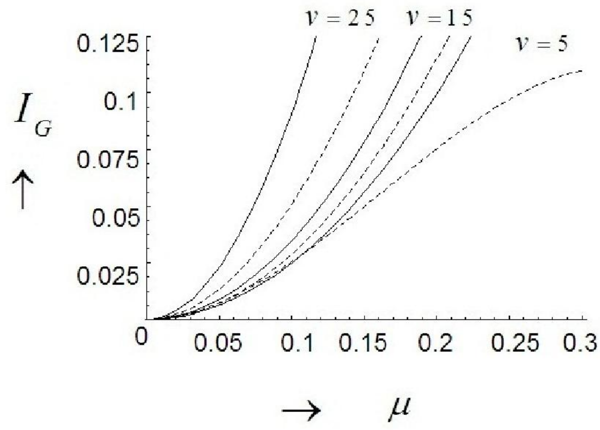

Fig. 3. Growth rate of instability I_G of the second wave train against the perturbation wave number μ for some different values of dimensionless wind velocity v written on the graphs. Here $\eta_{01} = 0.04, \quad \eta_{02} = 0.1, \quad (k_1, k_2) = (0.7132, 0.3394)$.---------------- represents fourth order results and .... represents third order results

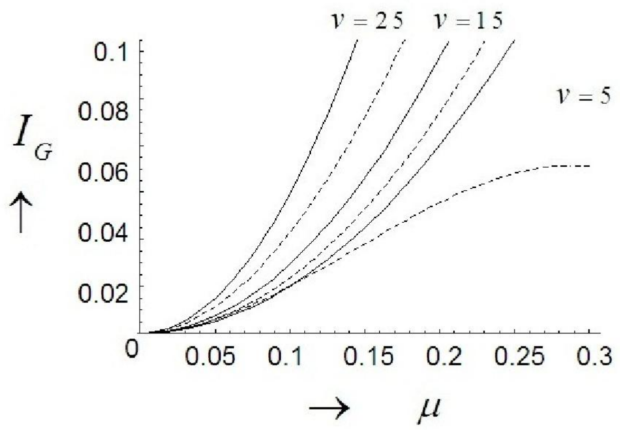

Fig. 4. Growth rate of instability I_G of the second wave train against the perturbation wave number μ for some different values of dimensionless wind velocity v written on the graphs. Here $\eta_{01} = 0.08, \quad \eta_{02} = 0.1, \quad (k_1, k_2) = (0.7132, 0.3394)$.----------------- represents fourth order results and .... represents third order results

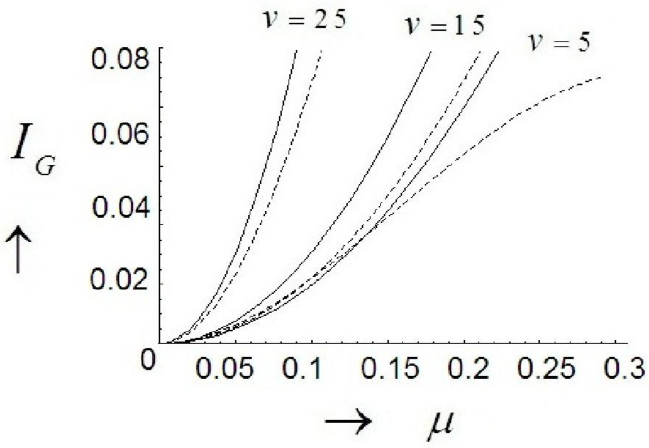

Fig. 5. Growth rate of instability I_G of the second wave train against the perturbation wave number μ for some different values of dimensionless wind velocity v written on the graphs. Here $\eta_{01} = 0, \quad \eta_{02} = 0.1, \quad (k_1, k_2) = (0.7132, 0.3394)$.----------------- represents fourth order results and .... represents third order results

DISCUSSION AND CONCLUSION

We have derived analytically two coupled fourth order nonlinear evolution equations in deep water for two gravity wave packets propagated in the same direction in the presence of wind flowing over water. The reason for starting from fourth order nonlinear evolution equation is motivated by the fact, as shown by Dysthe (1979) that a fourth order nonlinear evolution equation is a good starting point for making stability analysis of a uniform wave train in deep water. Here we have used a general method, based on Zakharov integral equation for the derivation of evolution equations. The two evolution equations are then used to investigate the stability analysis of a uniform surface gravity wave train in the presence of another gravity wave train when the group velocities of the two wave train in the presence of first wave train are plotted in figure 1 for different values of wind velocity. We have also plotted the marginal stability curves of the second wave train in the absence of the first wave train in figure 2 for different values of wind velocity.

It is found that the instability region of the second wave train expands due to the presence of the first wave train for fixed value of wind velocity. We also observe that the instability region is shortened slightly by the inclusion of fourth order terms. Further with the increase of wind velocity, the instability region is again shortened for fixed value of the amplitude of the first wave train. The growth rate of instability of the second wave train against the perturbation wave number have been plotted in figures 3, and 4 in the presence of first wave train and for different values of wind velocity. We have plotted in figure 5 the similar curves in the absence of first wave train and also have plotted the corresponding curves that can be obtained from third order evolution equations. From these figures, it is found that the growth rate of instability of the second wave train increases due to the presence of the first wave train and it increases with the increase of the amplitude of the first wave train for fixed value of wind velocity. We also observe that the influence of fourth order terms is to increase the growth rate of instability. Further the growth rate of instability increases with the increase of wind velocity for fixed value of the amplitude of the first wave train.

Appendix

$$\delta_1^{(1)} = \frac{m}{2(1+\gamma)n}\left[\gamma v + \frac{S_2^{(1)}}{(\gamma v + S_1^{(1)})}\right], \qquad \delta_2^{(1)} = \frac{m^2}{8(1+\gamma)n}\left[\frac{4S_1^{(1)}\gamma v^2 + S_2^{(1)2}}{S_1^{(1)2}}\right]$$

$$\delta_3^{(1)} = \frac{3m^3}{16(1+\gamma)n}\left[\frac{4S_1^{(1)}S_2^{(1)}\gamma v^2 - 8(1+\gamma)S_1^{(1)2}}{S_1^{(1)3}}\right], \qquad \delta_1^{(2)} = \frac{m}{2(1+\gamma)n}\left[\gamma v + \frac{S_2^{(2)}}{(\gamma v + S_1^{(2)})}\right]$$

$$\delta_2^{(2)} = \frac{m^2}{8(1+\gamma)n}\left[\frac{4S_1^{(2)}\gamma v^2 + S_2^{(2)2}}{S_1^{(2)2}}\right], \qquad \delta_3^{(2)} = \frac{-3m^3}{16(1+\gamma)n}\left[\frac{4S_1^{(2)}S_2^{(2)}\gamma v^2 - 8(1+\gamma)S_1^{(2)2}}{S_1^{(2)3}}\right]$$

$$\gamma_1^{(1)} = \frac{-3\sqrt{1+\gamma}\ m}{16n}\left[\frac{\gamma v + 7(1-\gamma^2) + S_1^{(1)} + \gamma v^2}{S_1^{(1)}(\gamma v + 1 - \gamma^2)}\right], \qquad \gamma_2^{(1)} = \frac{-3\sqrt{1+\gamma}\ m}{16n}\left[\frac{\gamma v + 7(1-\gamma^2) + S_1^{(1)}}{S_1^{(1)2}(\gamma v + 1 - \gamma^2)}\right]$$

$$\gamma_3^{(1)} = \frac{-\sqrt{1+\gamma}\ m}{32n}\left[\frac{\{\gamma v + 2(1-\gamma) - S_1^{(1)}\}\{\gamma v + 7(1-\gamma^2) + S_1^{(1)} + \gamma v^2}{S_1^{(1)2}(\gamma v + 1 - \gamma^2)}\right],$$

$$\gamma_4^{(1)} = \frac{\sqrt{1+\gamma}\ m}{2n}\left[1 + 2\gamma v - \gamma^2\right], \quad \gamma_1^{(2)} = \frac{\sqrt{1+\gamma}\ m}{16n}\left[\frac{\gamma v + 7(1-\gamma^2) + S_1^{(2)} + \gamma v^2}{S_1^{(2)}(\gamma v + 1 - \gamma^2)}\right],$$

$$\gamma_2^{(2)} = \frac{-3\sqrt{1+\gamma}\ m}{16n}\left[\frac{\gamma v + 7(1-\gamma^2) + S_1^{(2)}}{S_1^{(2)2}(\gamma v + 1 - \gamma^2)}\right], \quad \gamma_4^{(2)} = \frac{\sqrt{1+\gamma}}{2}\left[1 + 2\gamma v - \gamma^2\right]$$

$$\gamma_3^{(2)} = \frac{-\sqrt{1+\gamma}\ m}{32n}\left[\frac{\{\gamma v + 2(1-\gamma) - S_1^{(2)}\}\{\gamma v + 7(1-\gamma^2) + S_1^{(2)} + \gamma v^2}{S_1^{(2)2}(\gamma v + 1 - \gamma^2)}\right],$$

$$\beta_1^{(1)} = \frac{\sqrt{1+\gamma}}{8m^3}\left[8\gamma v + \gamma v^2 + 4m^2 - 2m(1-m)/n\right],$$

$$\beta_2^{(1)} = \frac{\sqrt{1+\gamma}}{16m^2}\left[2\gamma v - 3\gamma v^2 + 2m(1-m)n - 4mn - 16m^2 + 4m(1+m)n^2\right]$$

$$\beta_3^{(1)} = \frac{\sqrt{1+\gamma}}{16m^2}\left[5\gamma v - 7\gamma v^2 + 2m^2(3-\gamma v) - 2m(3 + 2\gamma v + (\gamma v - 3v^2)/5\right]$$

$$\beta_4^{(1)} = \frac{\sqrt{1+\gamma}}{8m^2}\left[\gamma v - 3\gamma v^2 + m(1-m) - m^2(3-\gamma v) + m(3 + 2\gamma v^2) - n(\gamma v^2 - 2m^2 n)\right]$$

$$\beta_5^{(1)} = \frac{\sqrt{1+\gamma}}{2m}(1 - 2\gamma v)$$

$$\beta_1^{(2)} = \frac{\sqrt{1+\gamma}}{8m^3}\left[8\gamma v + \gamma v^2 + 4m^2 - 2m(1-m)/n\right]$$

$$\beta_2^{(2)} = \frac{\sqrt{1+\gamma}}{16m^2}\left[2\gamma v - 3\gamma v^2 + 2m(1-m)n - 4mn - 16m^2 + 4m(1+m)n^2\right]$$

$$\beta_3^{(2)} = \frac{\sqrt{1+\gamma}}{16m^2}\left[5\gamma v - 7\gamma v^2 + 2m^2(3-\gamma v) - 2m(3 + 2\gamma v + (\gamma v - 3v^2)/5\right]$$

$$\beta_4^{(2)} = \frac{\sqrt{1+\gamma}}{8m^2}\left[\gamma v - 3\gamma v^2 + m(1-m) - m^2(3-\gamma v) + m(3 + 2\gamma v^2) - n(\gamma v^2 - 2m^2 n)\right]$$

$$\beta_5^{(2)} = \frac{\sqrt{1+\gamma}}{2n}(1 - 2\gamma v)$$

Where

$$S_1^{(1)} = (1-\gamma^2) - \gamma v^2 k_1, \quad S_2^{(1)} = (1-\gamma^2) - \gamma v^2 k_1,$$
$$S_1^{(2)} = (1-\gamma^2) - \gamma v^2 k_2, \quad S_2^{(2)} = (1-\gamma^2) - \gamma v^2 k_2$$

Nomenclature

$\left.\begin{array}{l}\delta_i^{(1)}(i=1,2,3)\\ \gamma_i^{(1)}(i=1,2,3,4)\\ \beta_i^{(1)}(i=1,2,3,4,5)\end{array}\right\}$ - coefficients given in the Appendix,

ε - slowness parameter,

α - wave steepness ,

ζ - elevation of the air water interface,

ω - frequency,

$T(k,k_1,k_2,k_3)$ -scalar function given first Zakharov,

γ - ratio of densities of air to water,

$\Delta\omega$ - frequency shift,

Ω - perturbed frequency at marginal stability.

(k,k_1,k_2,k_3) -wave vector,

g – acceleration due to gravity

H – Hilbert's transform operator,

μ - wave number,

s – dimensionless surface tension,

t – time,

v - air flow velocity,

I_G -growth rate of instability

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