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Review Article

PHYSICAL APPLICATIONS OF ISOTROPIC TENSOR

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ABSTRACT

Mathematically, physical quantities are presented in general by tensors. In this paper we treat only tensor, in the cartesian coordinate system, recognizing that space where defined tensorial space, obeys a euclidian geometry. We will not examine cases curved coordinate systems or spaces where is not euclidian geometry. We use the concept of isotropic tensor in continuum mechanics system. Using invariant to infinitely small rotations, we proved the original so popular as some statements: a) any second-ranking tensors T_{ij} can be written as $\lambda \ \ \delta_{ij}$; b) a isotropic tensor fourth ranking T_{ikmp} can be written in the form: $T_{ikmp} = \lambda \ \ \delta_{ik}\delta_{mp} + \mu \ \ (\delta_{im}\delta_{kp} + \delta_{ip}\delta_{km}) + \beta \ \ (\delta_{im}\delta_{kp} \quad \delta_{ip}\delta_{km})$. Given the general form of Hooke's law, and using the conclusion to isotropic tensor fourth ranking above, we draw the shape of Hooke's law for flexible environments isotrope.

INTRODUCTION

The concept of isotropy is used frequently as a simplifying assumption in continuum mechanics. Materials whose mechanical properties do not depend on directions are said to be isotropic. A material is isotropic if its constitutive equation (the stress-strain-history law) is unaltered under orthogonal transformations of coordinates. Orthogonal transformations consist of translations, rotations, and reflections of coordinate axes. The form of constitutive equation remain unchanged no matter how the axes are translated, rotated, or reflected. A tensor is said to be an isotropic tensor if its components remain invariant however the axes are rotated. An isotropic tensor possesses no directional properties. Therefore a non – zero vector (or a non – zero tensor of rank one) can never be an isotropic tensor. Tensors of higher orders, other than one, can be isotropic tensors. Tensors can describe different physical areas, from simple ones that are scalar fields as areas the temperature of vector fields such as flow velocity field of fluid or areas of deformation and strain, etc. In particular, we will study some important applications of isotropic tensors, trying to present in the original form any result that other authors have mentioned or draw on different routes.

ISOTROPIC TENSOR OF RANK TWO

Consider a general tensor T of rank two, with components T_{ij} with respect to some set of axes $\{e_1, e_2, e_3\}$.

In the infinitesimal rotations with an angle $d\boldsymbol{\theta}$, the elementar vector $d\boldsymbol{x}$ is:

$$d\boldsymbol{x} = d\boldsymbol{\theta} \times \boldsymbol{x}, \tag{2.1}$$

and dhe vector $\boldsymbol{x}$ becomes:

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{dx} = \boldsymbol{x} + d\boldsymbol{\theta} \times \boldsymbol{x}, \tag{2.2}$$

In the component form we have:

$$x'_i = (\delta_{ij} \quad \varepsilon_{kij} d\theta_k) x_j = a_{ij} x_j, \tag{2.3}$$

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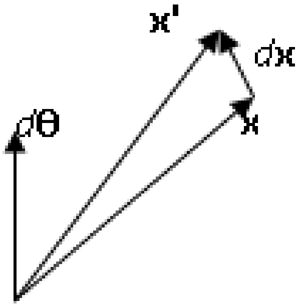


Fig. 1. Infinitesimal rotations

where

$$a_{ij} = \delta_{ij} \quad \varepsilon_{kij} d\theta_k = \begin{pmatrix} 1 & d\theta_3 & d\theta_2 \\ d\theta_3 & 1 & d\theta_1 \\ d\theta_2 & d\theta_1 & 1 \end{pmatrix} \tag{2.4}$$

Suppose that T is isotropic. Its components should then be unaltered by a rotation of 90° about the 3-axis, so $d\theta_1 = d\theta_2 = 0$ and $d\theta_3 = d\theta$, therefore:

$$a_{ij} = \delta_{ij} \quad \varepsilon_{kij} d\theta \tag{2.5}$$

For second rank tensors T_{ij}, the transformation law is:

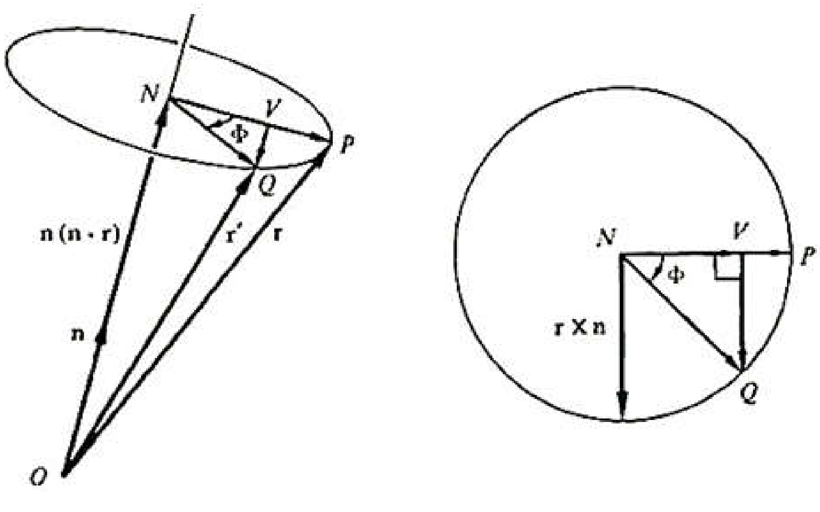
$$T_{ij} = a_{im} a_{jn} T_{mn} = T'_{ij} \tag{2.6}$$

Therefore we find:

$$\varepsilon_{3im} T_{mj} + \varepsilon_{3jn} T_{in} = 0 \text{ ose } \varepsilon_{kim} T_{mj} + \varepsilon_{kjn} T_{in} = 0, \tag{2.7}$$

and:

$$T_{11} = T_{22} = T_{33} \tag{2.8}$$



(a) Overall view **(b) The plane normal to the axis of rotation**

Figure 2.1. Vector diagrams for derivation of the rotation formula

The rotation formula is:

$$\boldsymbol{r}' = \boldsymbol{r} cos\Phi + \boldsymbol{n}(\boldsymbol{n} \quad \boldsymbol{r})(1 \quad cos\Phi) + (\boldsymbol{r} \times \boldsymbol{n}) sin\Phi \tag{2.9}$$

The four parameters $\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$ describing a finite rotation about an arbitrary axis. The Euler parameters are defined by

$$\boldsymbol{e}_o = cos\frac{\Phi}{2}\ ,\boldsymbol{e} = \begin{bmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\\ \boldsymbol{e}_3\end{bmatrix} = \boldsymbol{n}\ sin\frac{\Phi}{2}\ , \tag{2.10}$$

where $\boldsymbol{n}$ is the unit vector, and are a quaternion in scalar-vector representation

$$(e_0, e) = e_0 + e_1 i + e_2 j + e_3 k. \tag{2.11}$$

Because Euler's rotation theorem states that an arbitrary rotation may be described by only three parameters, a relationship must exist between these four quantities,

$$\boldsymbol{e}_0^2 + \boldsymbol{e}\quad \boldsymbol{e} = \boldsymbol{e}_0^2 + \boldsymbol{e}_1^2 + \boldsymbol{e}_2^2 + \boldsymbol{e}_3^2 = 1. \tag{2.12}$$

The rotation formula in terms of Euler parameters is

$$\boldsymbol{r}' = \boldsymbol{r}(\boldsymbol{e}_0^2 \quad \boldsymbol{e}_1^2 \quad \boldsymbol{e}_2^2 \quad \boldsymbol{e}_3^2) + 2\boldsymbol{e}(\boldsymbol{e}\quad \boldsymbol{r}) + (\boldsymbol{r} \times \boldsymbol{e})sin\Phi, \tag{2.13}$$

or

$$\begin{pmatrix}x'\\ y'\\ z'\end{pmatrix} = \boldsymbol{A}\begin{pmatrix}x\\ y\\ z\end{pmatrix}, \tag{2.14}$$

where the elements of matrix $\boldsymbol{A}$ are:

$$a_{ij} = \delta_{ij}(\boldsymbol{e}_0^2 \quad \boldsymbol{e}_k\quad \boldsymbol{e}_k) + 2\boldsymbol{e}_i\quad \boldsymbol{e}_j + 2\varepsilon_{ijk}\quad \boldsymbol{e}_0\quad \boldsymbol{e}_k. \tag{2.15}$$

Consider a general tensor T of rank two, with components T_{ij} with respect to some set of axes $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. Suppose that T is isotropic. Its components should then be unaltered by a rotation of 90° about the 3-axis, i.e., with respect to new axes

$$\boldsymbol{e'}_1 = \boldsymbol{e}_2, \quad \boldsymbol{e'}_2 = \quad \boldsymbol{e}_1, \quad \boldsymbol{e'}_3 = \boldsymbol{e}_3 \tag{2.16}$$

In this case the Euler parameters are:

$$\boldsymbol{e}_0 = cos45^0 = \frac{\sqrt{2}}{2}, \qquad \boldsymbol{e}_1 = 0, \qquad \boldsymbol{e}_2 = 0, \qquad \boldsymbol{e}_3 = \frac{\sqrt{2}}{2}, \tag{2.17}$$

And the matrix of this rotation is:

$$\boldsymbol{A} = \begin{pmatrix}0 & 1 & 0\\ \ \ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{2.18}$$

Using the matrix formulation of the transformation law for tensors, we see that

$$T_{13} = T_{23} = 0,\ T_{31} = T_{32} = 0. \tag{2.19}$$

Similarly, considering a rotation of 90° about the 2-axis, we find that

$$T_{12} = T_{32} = 0,\ T_{21} = T_{23} = 0 \tag{2.20}$$

Therefore all off-diagonal elements of T are zero, and all diagonal elements are equal, say λ. Thus

$$T_{ij} = \lambda\quad \delta_{ij} \tag{2.21}$$

In summary, we have shown that any isotropic second rank tensor must be equal to $\lambda\quad \delta_{ij}$ for some scalar λ.

ISOTROPIC TENSOR OF RANG FOUR

Other ways to characterize isotropy one may define the property of an elastic body through the strain energy function $U(\varepsilon_{ik})$ to relate the stress with strain components by,

$$\sigma_{ik} = \frac{\partial U}{\partial \varepsilon_{ik}} \tag{3.1}$$

where,

$$\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \text{ and } \hat{\sigma} = \frac{d\boldsymbol{F}}{d\boldsymbol{S}}, \tag{3.2}$$

The density of potential energy is:

$$U = c_0 + \left(\frac{\partial U}{\partial \varepsilon_{ik}}\right)_{\varepsilon_{ik}=0} \varepsilon_{ik} + \frac{1}{2}\left(\frac{\partial^2 U}{\partial \varepsilon_{ik}\partial \varepsilon_{jm}}\right)_{\varepsilon_{ik}=0} \varepsilon_{ik}\ \varepsilon_{jm} + \frac{1}{6}\left(\frac{\partial^3 U}{\partial \varepsilon_{ik}\partial \varepsilon_{jm}\varepsilon_{np}}\right)_{\varepsilon_{ik}=0} \varepsilon_{ik}\ \varepsilon_{jm}\ \varepsilon_{np} + \quad , \tag{3.3}$$

or

$$U = c_0 + \sigma_{ik}\ \varepsilon_{ik} + \frac{1}{2}C_{ikjm}\ \varepsilon_{ik}\ \varepsilon_{jm} + \frac{1}{6}C_{ikjmnp}\ \varepsilon_{ik}\ \varepsilon_{jm}\ \varepsilon_{np} + \quad , \tag{3.4}$$

where c_0 is a constant, C_{ikjm} is a tensor of order four and C_{ikjmnp} is a tensor of order six. For the symmetry of the tensors we have:

$$C_{ikjm} = \left(\frac{\partial^2 U}{\partial \varepsilon_{ik}\partial \varepsilon_{jm}}\right)_{\varepsilon_{ik}=0} = \left(\frac{\partial^2 U}{\partial \varepsilon_{jm}\partial \varepsilon_{ik}}\right)_{\varepsilon_{ik}=0} = C_{jmik} \tag{3.5}$$

So

$$C_{ikjm} = C_{jmik}. \tag{3.6}$$

If we multiply both sides of equation (3.2.3) with $d\varepsilon_{ik}$ then we have:

$$\sigma_{ik}\ d\varepsilon_{ik} = \frac{\partial U}{\partial \varepsilon_{ik}}\ d\varepsilon_{ik} = \frac{\partial^2 U}{\partial \varepsilon_{ik}\partial \varepsilon_{jm}}\varepsilon_{jm}\ d\varepsilon_{ik} = dU \tag{3.7}$$

So

$$dU = C_{ikjm}\ \varepsilon_{jm}\ d\varepsilon_{ik}, \tag{3.8}$$

Changing the dummy indices, we obtain

$$dU = C_{jmik}\ \varepsilon_{ik}\ d\varepsilon_{jm}\ , \tag{3.9}$$

But $C_{ikjm} = C_{jmik}$, therefor

$$dU = C_{ikjm}\ \varepsilon_{ik}\ d\varepsilon_{jm} \tag{3.10}$$

Adding Equations (3.2.8) and (3.2.10), we find

$$2dU = C_{ikjm}\left(\varepsilon_{jm}\ d\varepsilon_{ik} + \varepsilon_{ik}\ d\varepsilon_{jm}\right) = C_{ikjm}d\left(\varepsilon_{ik}\ \varepsilon_{jm}\right),$$

from which we obtain

$$U = \frac{1}{2}C_{ikjm}\ \varepsilon_{ik}\ \varepsilon_{jm}, \tag{3.11}$$

Where

$$C_{ikjm} = \left(\frac{\partial^2 U}{\partial \varepsilon_{ik}\partial \varepsilon_{jm}}\right), \tag{3.12}$$

A material is said to be isotropic if its mechanical properties can be described without reference to directions. When this is not true, the material is said to be anisotropic. Many structural metals such as steel and aluminum can be regarded as isotropic without appreciable error. We had, for a linearly elastic solid, with respect to the $\boldsymbol{e}_i$ basis,

$$\sigma_{ik} = C_{ikjm}\ \varepsilon_{jm}, \tag{3.13}$$

and with respect to the $\boldsymbol{e}'_i$ basis,

$$\sigma'_{ik} = C'_{ikjm}\ \varepsilon_{jm} \tag{3.13'}$$

If the material is isotropic, then the components of the elasticity tensor must remain the same, regardless of how the rectangular basis is rotated and reflected. That is,

$$C_{ikjm} = C'_{ikjm} \qquad (3.14)$$

If, in addition, the function is to be linear, then we have, in component form,

$$\begin{aligned}
\sigma_{11} &= C_{1111} \ \ \varepsilon_{11} + C_{1112} \ \ \varepsilon_{12} + \ldots\ldots + C_{1133} \ \ \varepsilon_{33}\,, \\
\sigma_{12} &= C_{1211} \ \ \varepsilon_{11} + C_{1212} \ \ \varepsilon_{12} + \ldots\ldots + C_{1233} \ \ \varepsilon_{33}\,, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\sigma_{33} &= C_{3311} \ \ \varepsilon_{11} + C_{3312} \ \ \varepsilon_{12} + \ldots\ldots + C_{3333} \ \ \varepsilon_{33}\,.
\end{aligned}$$

On the other hand we know that the law of conversion to a fourth rank tensor T_{ikmp}, written:

$$T'_{ikmp} = a_{ij}a_{kl}a_{mn}a_{pq}T_{jlnq} \qquad (3.15)$$

We can prove that if the relation (3.13) is isotropic, then T_{ikmp} is an isotropic tensor. Consider on infinitesimal rotations given by the transformation $x'_i = (\delta_{ij} \quad \varepsilon_{sij}d\theta)x_j = a_{ij}x_j$, where $a_{ij} = (\delta_{ij} \quad \varepsilon_{sij}d\theta)$. So that:

$$\begin{aligned}
T'_{ikmp} = (\delta_{ij} \quad \varepsilon_{sij}d\theta)\,(\delta_{kl} \quad \varepsilon_{skl}d\theta)(\delta_{mn} \quad \varepsilon_{smn}d\theta)\,(\delta_{pq} \quad \varepsilon_{spq}d\theta)T_{jlnq} = \\
= \delta_{ij}\delta_{kl}\delta_{mn}\delta_{pq}T_{jlnq} \quad \varepsilon_{spq}\delta_{ij}\delta_{kl}\delta_{mn}T_{jlnq}d\theta \quad \varepsilon_{smn}\delta_{ij}\delta_{kl}\delta_{pq}T_{jlnq}d\theta + \varepsilon_{smn}\varepsilon_{spq}\delta_{ij}\delta_{kl}T_{jlnq}d\theta^2 \\
\varepsilon_{skl}\delta_{ij}\delta_{mn}\delta_{pq}T_{jlnq}d\theta + \varepsilon_{skl}\varepsilon_{spq}\delta_{ij}\delta_{mn}T_{jlnq}d\theta^2 + \varepsilon_{skl}\varepsilon_{smn}\delta_{ij}\delta_{pq}T_{jlnq}d\theta^2 \quad \varepsilon_{skl}\varepsilon_{smn}\varepsilon_{spq}\delta_{ij}T_{jlnq}d\theta^3 \\
\varepsilon_{sij}\delta_{kl}\delta_{mn}\delta_{pq}T_{jlnq}d\theta + \varepsilon_{sij}\varepsilon_{spq}\delta_{kl}\delta_{mn}T_{jlnq}d\theta^2 + \varepsilon_{sij}\varepsilon_{smn}\delta_{kl}\delta_{pq}T_{jlnq}d\theta^2 \quad \varepsilon_{sij}\varepsilon_{smn}\varepsilon_{spq}\delta_{kl}T_{jlnq}d\theta^3 \\
+ \varepsilon_{sij}\varepsilon_{skl}\delta_{mn}\delta_{pq}T_{jlnq}d\theta^2 \quad \varepsilon_{sij}\varepsilon_{skl}\varepsilon_{spq}\delta_{mn}T_{jlnq}d\theta^3 \quad \varepsilon_{sij}\varepsilon_{skl}\varepsilon_{smn}\delta_{pq}T_{jlnq}d\theta^3 + \varepsilon_{sij}\varepsilon_{skl}\varepsilon_{smn}\varepsilon_{spq}T_{jlnq}d\theta^4,
\end{aligned}$$

Or

$$\begin{aligned}
&T'_{ikmp} = \\
&T_{ikmp} \quad d\theta\{\varepsilon_{spq}\delta_{ij}\delta_{kl}\delta_{mn}T_{jlnq} + \varepsilon_{smn}\delta_{ij}\delta_{kl}\delta_{pq}T_{jlnq} + \varepsilon_{skl}\delta_{ij}\delta_{mn}\delta_{pq}T_{jlnq} + \varepsilon_{sij}\delta_{kl}\delta_{mn}\delta_{pq}T_{jlnq}\} + d\theta^2\{\varepsilon_{smn}\varepsilon_{spq}\delta_{ij}\delta_{kl}T_{jlnq} + \\
&\varepsilon_{skl}\varepsilon_{spq}\delta_{ij}\delta_{mn}T_{jlnq} + \varepsilon_{skl}\varepsilon_{smn}\delta_{ij}\delta_{pq}T_{jlnq} + \varepsilon_{sij}\varepsilon_{spq}\delta_{kl}\delta_{mn}T_{jlnq} + \varepsilon_{sij}\varepsilon_{smn}\delta_{kl}\delta_{pq}T_{jlnq} + \varepsilon_{sij}\varepsilon_{skl}\delta_{mn}\delta_{pq}T_{jlnq}\} \\
&d\theta^3\{\varepsilon_{skl}\varepsilon_{smn}\varepsilon_{spq}\delta_{ij}T_{jlnq} + \varepsilon_{sij}\varepsilon_{smn}\varepsilon_{spq}\delta_{kl}T_{jlnq} + \varepsilon_{sij}\varepsilon_{skl}\varepsilon_{spq}\delta_{mn}T_{jlnq} + \varepsilon_{sij}\varepsilon_{skl}\varepsilon_{smn}\delta_{pq}T_{jlnq}\} + \varepsilon_{sij}\varepsilon_{skl}\varepsilon_{smn}\varepsilon_{spq}T_{jlnq}d\theta^4
\end{aligned}$$

By neglecting terms of second order and above, we get:

$$\begin{aligned}
T'_{ikmp} = T_{ikmp} \quad d\theta\{&\varepsilon_{spq}\delta_{ij}\delta_{kl}\delta_{mn}T_{jlnq} + \varepsilon_{smn}\delta_{ij}\delta_{kl}\delta_{pq}T_{jlnq} + \varepsilon_{skl}\delta_{ij}\delta_{mn}\delta_{pq}T_{jlnq} \\
&+ \varepsilon_{sij}\delta_{kl}\delta_{mn}\delta_{pq}T_{jlnq}\} \qquad (3.16)
\end{aligned}$$

On the other hand, we have that

$$\delta_{ij}\delta_{kl}\delta_{mn}T_{jlnq} = \delta_{ij}\delta_{kl}T_{jlmq} = \delta_{ij}T_{jkmq} = T_{ikmq} \qquad (3.17)$$

$$\delta_{ij}\delta_{kl}\delta_{pq}T_{jlnq} = \delta_{ij}\delta_{kl}T_{jlnp} = \delta_{ij}T_{jknp} = T_{iknp} \qquad (3.18)$$

$$\delta_{ij}\delta_{mn}\delta_{pq}T_{jlnq} = \delta_{ij}\delta_{mn}T_{jlnp} = \delta_{ij}T_{jlmp} = T_{ilmp} \qquad (3.19)$$

$$\delta_{kl}\delta_{mn}\delta_{pq}T_{jlnq} = \delta_{kl}\delta_{mn}T_{jlnp} = \delta_{kl}T_{jlmp} = T_{jkmp} \qquad (3.20)$$

Therefore:

$$T'_{ikmp} = T_{ikmp} \quad d\theta\{\varepsilon_{spq}T_{ikmq} + \varepsilon_{smn}T_{iknp} + \varepsilon_{skl}T_{ilmp} + \varepsilon_{sij}T_{jkmp}\} \qquad (3.21)$$

And knowing that we isotropic tensors $T'_{ikmp} = T_{ikmp}$, we find:

$$\varepsilon_{spq}T_{ikmq} + \varepsilon_{smn}T_{iknp} + \varepsilon_{skl}T_{ilmp} + \varepsilon_{sij}T_{jkmp} = 0 \qquad (3.22)$$

There are only three possible values for *i, k, m, p,* whence at least two of them must be equal. We may consider separately the cases where**(*a*)** two are equal and the other two unequal, **(*b*)** three are equal, **(*c*)** two are equal and the other two are equal, **(*d*)** all four are equal.

(a) for $i = k = 1, m = 2, p = 3$, equation (3.2.22) becomes:
(b)

$$\varepsilon_{s3q}T_{112q} + \varepsilon_{s2n}T_{11n3} + \varepsilon_{s1l}T_{1l23} + \varepsilon_{s1j}T_{j123} = 0 \qquad (3.23)$$

If we $n = 1,\ l = j = 2$ and $s = 3$, have:

$$\varepsilon_{33q}T_{112q} + \varepsilon_{321}T_{1113} + \varepsilon_{312}T_{1223} + \varepsilon_{312}T_{2123} = 0 \qquad (3.24)$$

The first term of equation (2.3.24) is zero anyway index value q because $\varepsilon_{33q} = 0$. Also $\varepsilon_{321} = \quad 1$, $\varepsilon_{312} = 1$, therefore equation (2.3.24) becomes:

$$T_{1113} + T_{1223} + T_{2123} = 0,$$

or,

$$T_{2123} + T_{1223} = T_{1113} \qquad (3.25)$$

Now we get $q = 1,\ l = j = 3$ and $s = 2$. So the equation (2.3.23) gives:

$$\varepsilon_{231}T_{1121} + \varepsilon_{22n}T_{11n3} + \varepsilon_{213}T_{1323} + \varepsilon_{213}T_{3123} = 0 \qquad (3.26)$$

The second term of (2.3.26) falls because $\varepsilon_{22n} = 0$, and knowing that $\varepsilon_{231} = 1$, $\varepsilon_{213} = \quad 1$, we get:

$$T_{1121} \quad T_{1323} \quad T_{3123} = 0,$$

$$T_{3123} + T_{1323} \quad T_{1121} = 0 \qquad (3.27)$$

Equation (3.2.25) and (3.2.27) put together in a system:

$$\begin{cases} T_{2123} + T_{1223} \quad T_{1113} = 0 \\ T_{3123} + T_{1323} \quad T_{1121} = 0 \end{cases} \qquad (3.28)$$

If in equation (2.3.23) put $s = 1, q = 2$ and $n = 3$, we have:

$$\varepsilon_{132}T_{1122} + \varepsilon_{123}T_{1133} + \varepsilon_{11l}T_{1l23} + \varepsilon_{11j}T_{j123} = 0 \qquad (3.29)$$

so that,

$$T_{1133} \quad T_{1122} = 0 \qquad (3.30)$$

Because $\varepsilon_{132} = \quad 1$, $\varepsilon_{123} = 1$, $\varepsilon_{11l} = 0$ and $\varepsilon_{11j} = 0$.

Other cases in point **a)** Can be obtained by exchanging the indices that are not already equal. Spare done in such a way as to bring 3 to 1, 1 to 2 and 2 to 3. So equation (3.2.30) gives:

$$T_{1122} = T_{1133} = T_{2233} = T_{2211} = T_{3322} = T_{3311} \qquad (3.31)$$

also,

$$T_{1212} = T_{1313} = T_{2323} = T_{2121} = T_{3232} = T_{3131} \qquad (3.32)$$
$$T_{2112} = T_{3113} = T_{3223} = T_{1221} = T_{2332} = T_{1331} \qquad (3.33)$$

(b) Take $i = k = m = 1$ and $p = 2$. In this case the relation (2.3.22), becomes:

$$\varepsilon_{s2q}T_{111q} + \varepsilon_{s1n}T_{11n2} + \varepsilon_{s1l}T_{1l12} + \varepsilon_{s1j}T_{j112} = 0, \qquad (3.34)$$

and if $s = 1, q = 3$, we have:

$$\varepsilon_{123}T_{1113} + \varepsilon_{11n}T_{11n2} + \varepsilon_{11l}T_{1l12} + \varepsilon_{11j}T_{j112} = 0 \qquad (3.35)$$

In equation (2.3.35), the term of the second, third and fourth term are zero whatever the values of the indices *n, l, j,* therefore:

$$T_{1113} = 0 \qquad (3.36)$$

Also, if the relation (2.3.34) put $s = 2,\ n = l = j = 3$ we find,

$$\varepsilon_{22q}T_{111q} + \varepsilon_{213}T_{1132} + \varepsilon_{213}T_{1312} + \varepsilon_{213}T_{3112} = 0, \qquad (3.37)$$

or,

$$0 \quad (T_{1132} + T_{1312} + T_{3112}) = 0,$$

from which we get:

$$T_{1132} + T_{1312} + T_{3112} = 0 \quad (3.38)$$

By replacing $T_{1113} = 0$ in the first equation of the system (2.3.28) we get,

$$T_{2123} + T_{1223} = 0 \quad (3.39)$$

Now we note $s = 3, q = 1$ and $n = l = j = 2$, so that equation (2.3.34) becomes:

$$\varepsilon_{321}T_{1111} + \varepsilon_{312}T_{1122} + \varepsilon_{312}T_{1212} + \varepsilon_{312}T_{2112} = 0,$$

or,

$$T_{1111} = T_{1122} + T_{1212} + T_{2112}, \quad (3.41)$$

Because $\varepsilon_{321} =\quad 1$ and $\varepsilon_{312} = 1$.

No further information is obtained by transforming components of **(*c*)** and **(*d*)**. Thus, if *i* = *k* = 1,*m* = *p* = 2, replacing *i* or *k* by *j* will yield a zero component unless *j* is equal to 1; then, the factorc_{ij} or $\mathbf{c_{kj}}$ is zero and the relation holds automatically. Similar considerations apply, if all of the *i, k,m, p* are equal. For $i = k = 1$ and $m = p = 2$ we have,

$$\varepsilon_{s2q}T_{112q} + \varepsilon_{s2n}T_{11n2} + \varepsilon_{s1l}T_{1l22} + \varepsilon_{s1j}T_{j122} = 0 \quad (3.42)$$

Take $s = 1$ and $q = n = 3$ so the above equation we have,

$$\varepsilon_{123}T_{1123} + \varepsilon_{123}T_{1132} + \varepsilon_{11l}T_{1l22} + \varepsilon_{11j}T_{j122} = 0 \quad (3.43)$$

We note that the last two terms in equation (3.2.43) are zero whatever the values of the indices l , j, so:

$$T_{1132} = \quad T_{1123} \quad (3.44)$$

If $s = 2$ and $l = j = 3$, then:

$$\varepsilon_{22q}T_{112q} + \varepsilon_{22n}T_{11n2} + \varepsilon_{213}T_{1322} + \varepsilon_{213}T_{3122} = 0 \quad (3.45)$$

From,

$$T_{3122} = \quad T_{1322} \quad (3.46)$$

If $s = 3$ and $q = n = l = j = 1$ we find:

$$\varepsilon_{321}(T_{1121} + T_{1112} + 0 + 0) \text{ or } T_{1121} = \quad T_{1112} \quad (3.47)$$

For $s = 3$, $q = n = 1$ and $l = j = 2$:

$$\varepsilon_{321}T_{1121} + \varepsilon_{321}T_{1112} + \varepsilon_{312}T_{1222} + \varepsilon_{312}T_{2122} = 0 \quad (3.48)$$

or,

$$\varepsilon_{321}(T_{1121} + T_{1112}) + \varepsilon_{312}(T_{1222} + T_{2122}) = 0$$

$$T_{1222} + T_{2122} = T_{1121} + T_{1112} \quad (3.49)$$

Another combination case **c)**; for $i = k = 2$ and $m = p = 1$:

$$\varepsilon_{s1q}T_{221q} + \varepsilon_{s1n}T_{22n1} + \varepsilon_{s2l}T_{2l11} + \varepsilon_{s2j}T_{j211} = 0 \quad (3.50)$$

If $s = 1$ dhe $l = j = 3$, then:

$$\varepsilon_{11q}T_{221q} + \varepsilon_{11n}T_{22n1} + \varepsilon_{123}T_{2311} + \varepsilon_{123}T_{3211} = 0 \quad (3.51)$$

As $\varepsilon_{11q} = \varepsilon_{11n} = 0$ and $\varepsilon_{123} = 1$, we find:

$$T_{2311} + T_{3211} = 0 \; ose \; T_{2311} = \quad T_{3211} \tag{3.52}$$

If $s = 2$ and $q = n = 3$:

$$\varepsilon_{213}T_{2213} + \varepsilon_{213}T_{2231} + \varepsilon_{22l}T_{2l11} + \varepsilon_{22j}T_{j211} = 0 \tag{3.53}$$

Whatever the values of indices *l, j* last two terms in equation (3.2.53) are zero, therefore:

$$T_{2213} + T_{2231} = 0 \; ose \; T_{2213} = \quad T_{2231} \tag{3.54}$$

If $s = 3$ and $q = n = l = j = 1$:

$$\varepsilon_{311}T_{2211} + \varepsilon_{311}T_{2211} + \varepsilon_{321}T_{2111} + \varepsilon_{321}T_{1211} = 0 \tag{3.55}$$

Where we,

$$T_{2111} + T_{1211} = 0 \; ose \; T_{2111} = \quad T_{1211} \tag{3.56}$$

Settings $s = 3$ dhe $q = n = l = j = 2$:

$$\varepsilon_{312}T_{2212} + \varepsilon_{312}T_{2221} + \varepsilon_{322}T_{2211} + \varepsilon_{322}T_{2211} = 0, \tag{3.57}$$

and we find,

$$T_{2212} + T_{2221} = 0 \; ose \; T_{2212} = \quad T_{2221} \tag{3.58}$$

The other case within case **c)** is the combination ; $i = k = 2$ and $m = p = 3$, so that by (2.3.22) we get:

$$\varepsilon_{s3q}T_{223q} + \varepsilon_{s3n}T_{22n3} + \varepsilon_{s2l}T_{2l33} + \varepsilon_{s2j}T_{j233} = 0 \tag{3.59}$$

Settings $s = 1$ and $q = n = 2$, $l = j = 3$, so that (2.3.59) becomes:

$$\varepsilon_{132}T_{2232} + \varepsilon_{132}T_{2223} + \varepsilon_{123}T_{2333} + \varepsilon_{123}T_{3233} = 0 \tag{3.60}$$

$$T_{2232} \quad T_{2223} + T_{2333} + T_{3233} = 0,$$

$$T_{2333} + T_{3233} = T_{2232} + T_{2223} \tag{3.61}$$

Settings $s = 1$ and $q = n = 2$, $l = j = 2 \; ose \; l = j = 1$, and have:

$$\varepsilon_{132}T_{2232} + \varepsilon_{132}T_{2223} + \varepsilon_{121}T_{2133} + \varepsilon_{121}T_{1233} = 0, \tag{3.62}$$

or

$$T_{2232} + T_{2223} = 0 \text{ ose } T_{2232} = \quad T_{2223} \tag{3.63}$$

If $l = j = 3$, we have:

$$\varepsilon_{132}T_{2232} + \varepsilon_{132}T_{2223} + \varepsilon_{123}T_{2333} + \varepsilon_{123}T_{3233} = 0 \tag{3.64}$$

$$T_{2333} + T_{3233} = T_{2232} + T_{2223} \tag{3.65}$$

Settings $s = 2$ and $q = n = 1$:

$$\varepsilon_{231}T_{2231} + \varepsilon_{231}T_{2213} + \varepsilon_{22l}T_{2l33} + \varepsilon_{22j}T_{j233} = 0 \tag{3.66}$$

Last fall two terms, thus:

$$T_{2231} + T_{2213} = 0 \text{ or } T_{2231} = \quad T_{2213} \tag{3.67}$$

Settings $s = 3$ and $l = j = 1$:

$$\varepsilon_{33q}T_{223q} + \varepsilon_{33n}T_{22n3} + \varepsilon_{321}T_{2133} + \varepsilon_{321}T_{1233} = 0 \qquad (3.68)$$

The first two fall terms, therefore:

$$T_{2133} + T_{1233} = 0 \text{ ose } T_{2133} = \quad T_{1233} \qquad (3.69)$$

Combining the latest on this case is, for $i = k = 3$ and $m = p = 2$:

$$\varepsilon_{s2q}T_{332q} + \varepsilon_{s2n}T_{33n2} + \varepsilon_{s3l}T_{3l22} + \varepsilon_{s3j}T_{j322} = 0 \qquad (3.70)$$

Settings $s = 1$ and pick $q = n = 3, l = j = 2$:

$$\varepsilon_{123}T_{3323} + \varepsilon_{123}T_{3332} + \varepsilon_{132}T_{3222} + \varepsilon_{132}T_{2322} = 0, \qquad (3.71)$$

and we find,

$$T_{3323} + T_{3332} = T_{3222} + T_{2322} \qquad (3.72)$$

$s = 1$ and $q = n = 3$, $l = j = 1$ ose $l = j = 2$:

$$\varepsilon_{123}T_{3323} + \varepsilon_{123}T_{3332} + \varepsilon_{131}T_{3122} + \varepsilon_{131}T_{1322} = 0, \qquad (3.73)$$

from which we get.

$$T_{3323} = \quad T_{3332} \qquad (3.74)$$

Settings $s = 2$ and $l = j = 1$:

$$\varepsilon_{22q}T_{332q} + \varepsilon_{22n}T_{33n2} + \varepsilon_{231}T_{3122} + \varepsilon_{231}T_{1322} = 0 \qquad (3.75)$$

The first two terms are zero whatever the values of the indices q and n, therefore:

$$T_{3122} = \quad T_{1322} \qquad (3.76)$$

Settings $s = 3$ and $q = n = 1$:

$$\varepsilon_{321}T_{3321} + \varepsilon_{321}T_{3312} + \varepsilon_{33l}T_{3l22} + \varepsilon_{33j}T_{j322} = 0 \qquad (3.77)$$

The second two terms are zero whatever the values of indices *l* and *j*, therefore:

$$T_{3321} = \quad T_{3312} \qquad (3.78)$$

In the case of the four indices have equal , $i = k = m = p$. There are three cases:

1. For $i = k = m = p = 1$:
2.

$$\varepsilon_{s1q}T_{111q} + \varepsilon_{s1n}T_{11n1} + \varepsilon_{s1l}T_{1l11} + \varepsilon_{s1j}T_{j111} = 0 \qquad (3.79)$$

For $s = 1$ do not get any kind of information whatever the indices q, n, j, l For all terms fall.

Take $s = 2$ and $q = n = l = j = 3$:

$$\varepsilon_{213}T_{1113} + \varepsilon_{213}T_{1131} + \varepsilon_{213}T_{1311} + \varepsilon_{213}T_{3111} = 0, \qquad (3.80)$$

$$T_{1113} + T_{1131} + T_{1311} + T_{3111} = 0 \qquad (3.81)$$

Take $s = 3$ dhe $q = n = l = j = 2$:

$$\varepsilon_{312}T_{1112} + \varepsilon_{312}T_{1121} + \varepsilon_{312}T_{1211} + \varepsilon_{312}T_{2111} = 0, \qquad (3.82)$$

$$T_{1112} + T_{1121} + T_{1211} + T_{2111} = 0 \qquad (3.83)$$

3. For $i = k = m = p = 2$.

$$\varepsilon_{s2q}T_{222q} + \varepsilon_{s2n}T_{22n2} + \varepsilon_{s2l}T_{2l22} + \varepsilon_{s2j}T_{j222} = 0 \tag{3.84}$$

For $s = 2$ do not get information whatever the indices q, n, j, l for all terms fall.

Settings $s = 2$; also q, n, l, j should be different from 1 and 2, therefore, when we $q = n = l = j = 3$:

$$\varepsilon_{123}T_{2223} + \varepsilon_{123}T_{2232} + \varepsilon_{123}T_{2322} + \varepsilon_{123}T_{3222} = 0, \tag{3.85}$$

$$T_{2223} + T_{2232} + T_{2322} + T_{3222} = 0 \tag{3.86}$$

Take $s = 3$ and $q = n = l = j = 1$:

$$\varepsilon_{321}T_{2221} + \varepsilon_{321}T_{2212} + \varepsilon_{321}T_{2122} + \varepsilon_{321}T_{1222} = 0, \tag{3.87}$$

$$T_{2221} + T_{2212} + T_{2122} + T_{1222} = 0 \tag{3.88}$$

4. For $i = k = m = p = 3$.

$$\varepsilon_{s3q}T_{333q} + \varepsilon_{s3n}T_{33n3} + \varepsilon_{s3l}T_{3l33} + \varepsilon_{s3j}T_{j333} = 0 \tag{3.89}$$

For $s = 3$ do not get any kind of information whatever the indices *q, n, j, l* for all terms fall.
Settings $s = 1$; $q = n = l = j = 3$:

$$\varepsilon_{132}T_{3332} + \varepsilon_{132}T_{3323} + \varepsilon_{132}T_{3233} + \varepsilon_{132}T_{2333} = 0, \tag{3.90}$$

$$T_{3332} + T_{3323} + T_{3233} + T_{2333} = 0 \tag{3.91}$$

Take $s = 2,\ q = n = l = j = 1$ and we find:

$$T_{3331} + T_{3313} + T_{3133} + T_{1333} = 0 \tag{3.92}$$

We can denote the components of the equation (3.2.31) with λ,

$$T_{1122} = \lambda, \tag{3.93}$$

Those (3.2.32) the mark with $(\mu + \nu)$,

$$T_{1212} = (\mu + \nu), \tag{3.94}$$

and those of (3.2.20) the mark with $(\mu \quad \nu)$,

$$T_{2112} = (\mu \quad \nu), \tag{3.95}$$

Replacing (3.2.93), (3.2.94) and (3.2.95) in equation (3.2.41) we find:

$$T_{1111} = \lambda + (\mu + \nu) + (\mu \quad \nu) = \lambda + 2\mu \tag{3.96}$$

So we can write:

$$T_{1111} = T_{2222} = T_{3333} = \lambda + 2\mu \tag{3.97}$$

$T_{ikmp} = 1$ if $i = k$ dhe $m = p$. In this case T_{ikmp} has only one component and is expressed as the product of two second rank tensors $\delta_{ik}\delta_{mp}$.

$T_{ikmp} = 1$ if $i = m$ and $k = p, or$ if $i = p$, $k = m$ and $i \neq k$. Also if $i = k$ then gets to have two components for all other components are zero and it can be written:

$$T_{ikmp} = \delta_{im}\delta_{kp} + \delta_{ip}\delta_{km} \tag{3.98}$$

$T_{ikmp} = 1$ if $i = m$ and $k = p,$ and $T_{ikmp} = \quad 1$ if $i = p$ and $k = m$; and if $i = k, T_{ikmp} = 0$. For this reason tensor T_{ikmp} can be written as follows:

$$T_{ikmp} = \delta_{im}\delta_{kp} \quad \delta_{ip}\delta_{km} \,, \tag{3.99}$$

or,

$$T_{ikmp} = \varepsilon_{ijk}\varepsilon_{mjp} \tag{3.100}$$

Since, if $i = 1, j = 2$ and $k = 3$ have $\varepsilon_{123} = 1$ and if $= 3, j = 2$ and $k = 1$, we get $\varepsilon_{321} = \quad 1$. So,

$$T_{1313} = 1 \text{ and } T_{1331} = \quad 1, \tag{3.101}$$

with corresponding values for the other components. Evidently, (3.99) and (3.100) represent a tensor of order 4.
Finally, the general isotropic tensor of order 4 is therefore

$$T_{ikmp} = \lambda \;\; \delta_{ik}\delta_{mp} + \mu \;\; \left(\delta_{im}\delta_{kp} + \delta_{ip}\delta_{km}\right) + \beta \;\; \left(\delta_{im}\delta_{kp} \quad \delta_{ip}\delta_{km}\right), \tag{3.102}$$

where λ, μ and β are scalars.

CONCLUSION

If an elastic solid is isotropic, the tensor T_{ikmp} must be isotropic.

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