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Research Article

FREE MOTION OF AN ASYMETRIC GYROSTAT STUDIED IN THE PHASE PLANE

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ABSTRACT

The motion of the asymmetric gyrostat consisting of an asymmetrical carrier and an axisymmetric rotor, rotating about a fixed point under the action of the gravitational force. If the rotor of the gyrostat is locked, it does not have any effect on the dynamic behavior of the gyrostat. For purposes of investigating the effect of the rotor on the motion of the gyrostat, Deprit's canonical are introduced to establish the hamiltonian structure for this problem. Motion equations of the free gyrostat with small rotor asymmetry and small internal moment are obtained in Anoyer-Deprit variables. Control law for the internal moment is proposed that eliminates the possibility of the separatrix chaos. Numerical simulation shows the efficiency of the proposed control. When the coefficient of the gravitational torque is zero, the problem reduces to torque-free motion of the gyrostat. The torque-free motion of the asymmetrical gyrostat may then be described by a one-degree of freedom Hamiltonian system. In this paper we have studied the movement and stability of an asymmetric gyrostat in the phase plane. We have used linear stability analysis to determine the stability of equilibrium of the gyrostat. The number of equilibria changes as angular momentum is varied.

INTRODUCTION

The problems of the dynamics of the rotational motion of a gyrostat are very important for numerous applications such as the dynamics of satellite gyrostat, spacecraft, aircraft, robotics and the like. The attitude motion of the rigid body and the gyrostat has been extensively studied over a long period. In recent years, chaotic motions of the rigid body and the gyrostat under various circumstances have attracted interest of many researchers. Both numerical and analytical methods have been employed. The earlier studies on the attitude motion of a gyrostat could be found in the papers by Volterra. Volterra solved analytically the ordinary differential equations describing the attitude motions of a gyrostat with constant angular momentum and under no external torques. Rumyantsev reviewed the non-linear motion stability of liquid-filled solid bodies using Lyapunov's methods. When an inviscid, homogeneous, and incompressible liquid fills a cavity of the body completely and is subject to a body force derivable from a potential in irrotational motion, the ordinary differential equations of the liquid- filled body are identical to those of a solid body to which a rotating gyroscope is joined. If the rotor of the gyrostat is locked, it does not have any effect on the dynamic behavior of the gyrostat. For purposes of investigating the effect of the rotor on the motion of the gyrostat, Deprit's canonical are introduced to establish the hamiltonian structure for this problem. The effect of the rotor on the global motion of the gyrostat is studied by numerical simulation in conjunction with the poincare map. Kinsey et al. focused upon the capture dynamics of the precession phase lock, a phenomenon that could prevent the successful despin of a dual-spin spacecraft by developing a control strategy that employed closed-loop feedback control of the motor torque when the system was near resonance. Or studied the non-linear dynamics of an asymmetrical gyrostat and provided a numerical scheme for computing multiple equilibrium solutions and determining their stability and bifurcation properties simultaneously for equations that did not necessarily possess a potential. Hall investigated the escape from gyrostat trap statesa and proposed a procedure based upon the global analysis of the rotational dynamics.

EQUATION OF MOTION IN TERMS OF THE DEPRIT'S CANONICAL VARIABLES

The motion of the asymmetric gyrostat consisting of an asymmetrical carrier and an axisymmetric rotor, rotating about a fixed point under the action of the gravitational force.

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Consider a gyrostat, consiting af an asymmetrical carrier and an axisymmetric rotor (see Figure 1). The rotor rotates about a principal axis of the carrier. The Lagrangian of the system is:

$$= \frac{1}{2}\left(A\omega_x^2 + B\omega_y^2 + C\omega_z^2\right) + C_2\omega_z\Omega_z + \frac{1}{2}C_2\Omega_z^2 \quad mg(x_0 sin\theta sin\psi + y_0 sin\theta cos\psi + z_0 cos\theta) \tag{1}$$

where $A = A_1 + A_2$, $B = B_1 + A_2$, $C = C_1 + C_2$ and A_i, B_i, C_1 $(i = 1,2)$ are the principal inertia moments of the carrier and the rotor, respectively; ω_x, ω_y and ω_z are the components of the angular velocity $\bar{\omega}$ of the system in the body-fixed reference frame $(Oxyz)$, Ω_z is the relative spin speed of the rotor with respect to the carrier about the Oz axis. Here assume that the bearing of the rotor is frictionless, so that the relative angular momentum $L_z = C_2\Omega_z$ is a constant.

The angels θ, ϕ dhe ψ are the Euler angels and x_0, y_0, z_0 are the coordinates of the mass center of the gyrostat in the body-fixed reference frame $(Oxyz)$. In the terms of the Euler angles the angular velocity is $\bar{\omega} = \bar{\phi} + \theta + \bar{\psi}$ and the component of the angular velocity may also be expressed in terms of the Euler angles.

The angular momenta canonically conjugate to the velocities θ, ϕ dhe ψ are defined by

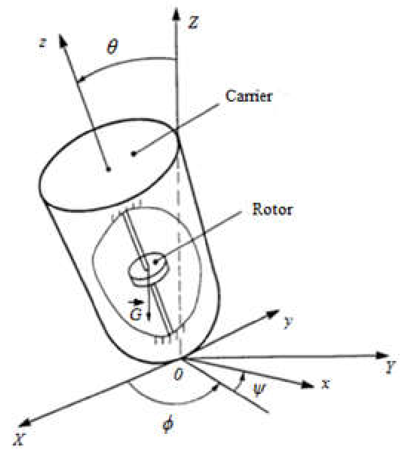

Figure 1. The asymmetric gyrostat

$$p_\phi = \frac{\partial}{\partial\phi} = \{A\,\omega_x\, sin\psi + B\omega_y cos\psi\}sin\theta + \{C\omega_z + \quad _z\}cos\theta \tag{2}$$

$$p_\theta = \frac{\partial}{\partial\theta} = A\omega_x\, cos\psi \quad B\omega_y sin\psi \tag{3}$$

$$p_\psi = \frac{\partial}{\partial\psi} = C\omega_z + \quad _z \tag{4}$$

and the Hamiltonian of the system is

$$= \tfrac{1}{2}\left(A\omega_x^2 + B\omega_y^2 + C\omega_z^2\right) + U(\theta, \psi) \tag{5}$$

where,$U(\theta, \psi) = (x_0 sin\theta sin\psi + y_0 sin\theta cos\psi + z_0 cos\theta)$ is the potential energy due to the gravitational force.

Consider two frames reference *OXYZ* and *Oxyz*. Let *ON* be the intersection of the plane *OXY* and *Oxy*. Then denote by ϕ the angle *XON*, by ψ the angle *NOx*, and by θ inclination of the plane *Oxy* on the plane *OXY*.

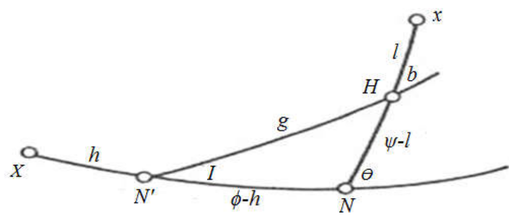

Figure 2. The frame of reference *Oxyz* is located with spect to the intermediate plane *ON'H* by angles *I* and *g*, while the plane *ON'H* is located with respect to the frame of reference *OXYZ* by the angles *h* and *I*

Now take a plane going through *O* which cuts the plane *Oxy* along the line *OH*, and the plane *OXY* along the line *ON'*. Let *h* denote the angle *XON'*; *g* the angle *N'OH*; and *l*, the angle *HOx*. Also let *I* be the inclination of the plane *ON'H* on the plane *OXY*; and *b*, the inclination of the plane *Oxy* on the plane *ON'H*.

The various angles just defined are represented in figure 2. The plan is to define a transformation $(\phi,\theta,\psi,p_\phi,p_\theta,p_\psi) \rightarrow$ (*l,g,h,L,G,H*) of the phase space of the three moments (p_ϕ,p_θ,p_ψ) into the phase space of the three coordinates (*l,g,h*) and the three momenta (*L,G,H*).

To this effect, first assume the relations

$$H = GcosI,\ L = Gcosb \tag{6}$$

Define the transformation of momenta by relations

$$p_\phi = H \tag{7}$$

$$p_\theta = Gsinbsin(l \quad \psi) \tag{8}$$

$$p_\psi = L \tag{9}$$

Finally, suppose that the angles ϕ , θ and $\psi \quad l$ are determined by the angles I, θ and *b* from the usual identities of spherical trigonometry applied to the spherical triangle *N'HN*.

The Hamiltonian of the system in terms of Deprit's variables is:

$$= \frac{1}{2}\left(\frac{sin^2 l}{A} + \frac{cos^2 l}{B}\right)(G^2 \quad L^2) + \frac{(L-h_z)^2}{2C} + \frac{x_0}{r_0}[sinI\ sing\ cosl + (cosI\ sinb + sinI\ cosb\ cosg)sinl] + \frac{y_0}{r_0}[cosI\ sinb + sinI\ cosb\ cosg)cosl \quad sinI\ sing\ sinl] + \frac{z_0}{r_0}(cosI\ cosb \quad sinI\ sinb\ cosg)\mu \tag{10}$$

where

$$\mu = mgr_0 = mg\sqrt{x_0^2 + y_0^2 + z_0^2} \tag{11}$$

is the coefficient of the gravitational torque.

STABILITY OF THE FREE ROTATION OF THE GYROSTAT

When $\mu = 0$, the problem reduces to torque-free motion of the gyrostat, so the Hamiltonian of the free rotations.

$$H_1 = \frac{1}{2}\left(\frac{sin^2 l}{A} + \frac{cos^2 l}{B}\right)(G^2 \quad L^2) + \frac{(L-h_z)^2}{2C} \tag{12}$$

is obtained from (10) by replacing $\mu = 0$.

Three of the phase variables are ignorable, and therefore, as was done in a previous instance, replaced by dashes, namely: (a) *H*, so that the longitude *h* of the node of the invariable plane in fixed reference plane *OXY* is a constant; (b) *g*, so that norm *G* of the angular momentum is a constant; (c) *h*, so that the component *H* of *G* is a constant. Now that *G* and *H* are both constant, we have, in view (6), that the inclination *I* of the invariable plane on the fixed reference plane *OXY* is constant. Therefore, the invariable plane is fixed in space. Hence the direction of *G* is fixed in space and, since it has also a constant norm the vector ***G*** itself is fixed in space.

The torque-free motion of the asymmetrical gyrostat may then be describe by a one-degree of freedom Hamiltonian system with the following equation of motion:

$$l = \frac{\partial \quad _1}{\partial L} = \left(\frac{1}{C} \quad \frac{sin^2 l}{A} \quad \frac{cos^2 l}{B}\right)L \quad \frac{h_z}{C} \tag{13}$$

$$L = \quad \frac{\partial \quad _1}{\partial l} = \left(\frac{1}{B} \quad \frac{1}{A}\right)(G^2 \quad L^2)sinl\ cosl \tag{14}$$

We use the linear stability analysis to find the stability of equilibria. So, the firs we have to solve the system of equations

$$\begin{cases} l = f(l, L) = 0 \\ L = g(l, L) = 0 \end{cases} \tag{15}$$

to find the fixed points. The Hamiltonian (30) is a periodic function of l; thus the dynamically significant part of the phase plane (l, L) is limited to the rectangle defined by $\quad G \leq L \leq G$ and by inequality $0 \leq l \leq \pi$.

There are three kinds of equilibria described by equations (13) and (14). For $l = 0, \pi$ we have the firs equilibrium:

$$l = 0 = \left(\frac{1}{C} \quad \frac{1}{B}\right) L \quad \frac{z}{C} = 0$$

or

$$L = \left(\frac{B}{B-C}\right) \quad z \tag{16}$$

The second equilibrium is for $l = \pi/2$ and $l = 3\pi/2$, so we have:

$$L = \left(\frac{A}{A-C}\right) \quad z \tag{17}$$

The third equilibrium is arbitrar, $|L| = G$.

From the physical requirement that $|L| \leq G$, we obtain the following conditions for the existence of the first equilibrium:

$$L = \left(\frac{B}{B-C}\right) \quad z \leq G \text{ or } \quad z \leq \left(1 \quad \frac{C}{B}\right) G \text{ if } B > C \tag{18}$$

$$L = \left(\frac{B}{B-C}\right) \quad z \geq \quad G \text{ or } \quad z \leq \left(\frac{C}{B} \quad 1\right) G \text{ if } B < C \tag{19}$$

For the existence of the second equilibrium we have:

$$L = \left(\frac{A}{A-C}\right) \quad z \leq G \text{ or } \quad z \leq \left(1 \quad \frac{C}{A}\right) G \text{ if } A > C \tag{20}$$

$$L = \left(\frac{A}{A-C}\right) \quad z \geq \quad G \text{ or } \quad z \leq \left(\frac{C}{A} \quad 1\right) G \text{ if } A < C \tag{21}$$

For the first equilibrium the fixed point is the point with coordinate $(l, L) = \left(0, \left(\frac{B}{B-C}\right) \quad z\right)$ and for the second equilibrium the fixed point is $(l, L) = \left(\frac{\pi}{2}, \left(\frac{A}{A-C}\right) \quad z\right)$.

The Jacobian of the system in the fixed point of the first equilibrium is:

$$A_{E_1} = \begin{pmatrix} 0 & \left(\frac{1}{C} \quad \frac{1}{B}\right) \\ \left(\frac{1}{B} \quad \frac{1}{A}\right)(G^2 \quad L^2) & 0 \end{pmatrix} \tag{22}$$

In general, the eigenvalues of a matrix A are given by the characteristic equation det $(A \quad \lambda I) = 0$, where I is the identity matrix. For a 2×2 matrix the solutions of the charasterictic equation are the eigenvalues

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} \tag{23}$$

where $\tau = tr(M)$ and $\Delta = detM$. For the matrix A_{E_1} in (22) we have

$$\tau = 0 \tag{24}$$

$$\Delta = \quad \left(\frac{1}{C} \quad \frac{1}{B}\right)\left(\frac{1}{B} \quad \frac{1}{A}\right)\left[G^2 \quad \left(\frac{B}{B-C} \quad z\right)^2\right] = \quad \left(\frac{1}{C} \quad \frac{1}{B}\right)\left(\frac{1}{B} \quad \frac{1}{A}\right)(G^2 \quad L^2) \tag{25}$$

$$\lambda_{1,2} = \pm\sqrt{\left(\frac{1}{C} \quad \frac{1}{B}\right)\left(\frac{1}{B} \quad \frac{1}{A}\right)(G^2 \quad L^2)} \tag{26}$$

For $A > B > C$ and $|L| \leq G$ we have $\Delta < 0$, so the eigenvalues are real and have opposite signs; hence the fixed point $\left(0, \left(\frac{B}{B-C}\right) \quad _z\right)$ is a *saddle point*, so the first equilibrium is unstable (saddle).

For the second equilibrium we have

$$A_{E_2} = \begin{pmatrix} 0 & \left(\frac{1}{C} \quad \frac{1}{A}\right) \\ 0 & 0 \end{pmatrix} \tag{27}$$

In this case, $\tau = 0$ and $\Delta = 0$, so the second equilibrium is stable. The third equilibrium various the angular momentum.

RESULTS AND CONCLUSIONS

The stability of the first and the second equilibrium depends only on the parameters *A*, *B*, *C*. Computations were performed for a gyrostat with the principal moments of inertia *A* = 1.2, *B* = 0.9 and *C* = 0.45.

Other parameters used are: the angular momentum h_z=1.9, the Hamiltonian H_0=55, *L*/*G*=0.4. Sabstituing this parameters into (12), we obtain G=11.3982. So the matrix A_{E_1} is

$$A_{E_1} = \begin{pmatrix} 0 & 1.1111 \\ 30.3144 & 0 \end{pmatrix} \tag{28}$$

The eigenvalues of this matrix are: $\lambda_{1,2} \pm 5.8036$, so they are real and have opposite signs; hence the fixed point

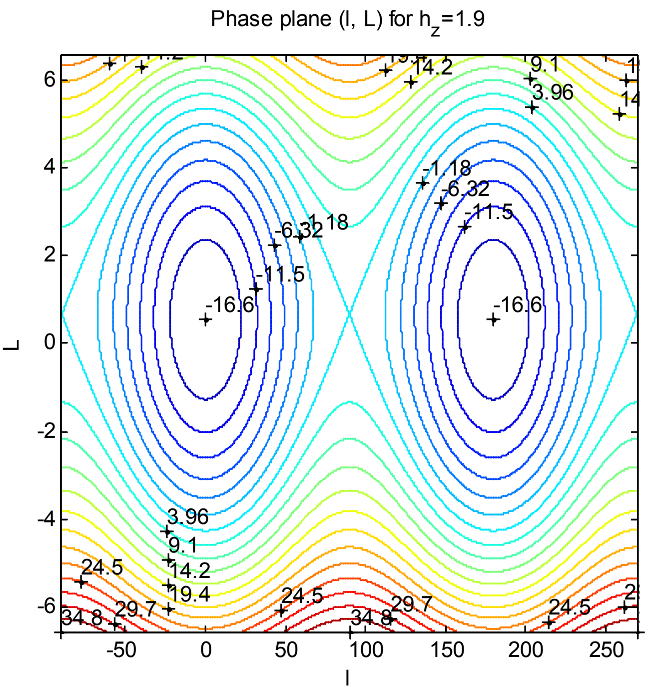

Figure 3. Isoenergetic curves in the phase plane (*l*,*L*) for the torque-free rotation of the gyrostat for $h_z = 1.9$

$\left(0, \left(\frac{B}{B-C}\right) \quad _z\right) = (0, 5.4)$ is a *saddle point*.

Therefore the first equilibrium is unstable (saddle). For the second equilibrium we have

$$A_{E_2} = \begin{pmatrix} 0 & 1.3889 \\ 0 & 0 \end{pmatrix} \tag{29}$$

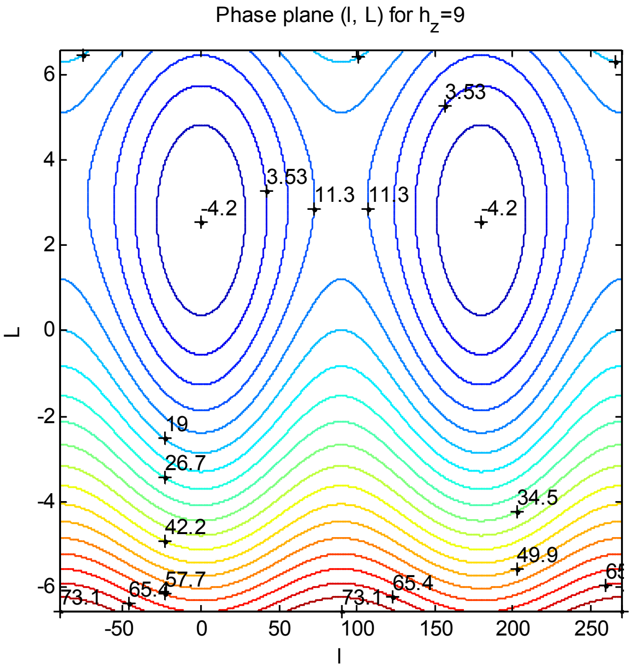

Figure 4. Isoenergetic curves in the phase plane (*l*,*L*) for the torque-free rotation of the gyrostat for $h_z = 9$

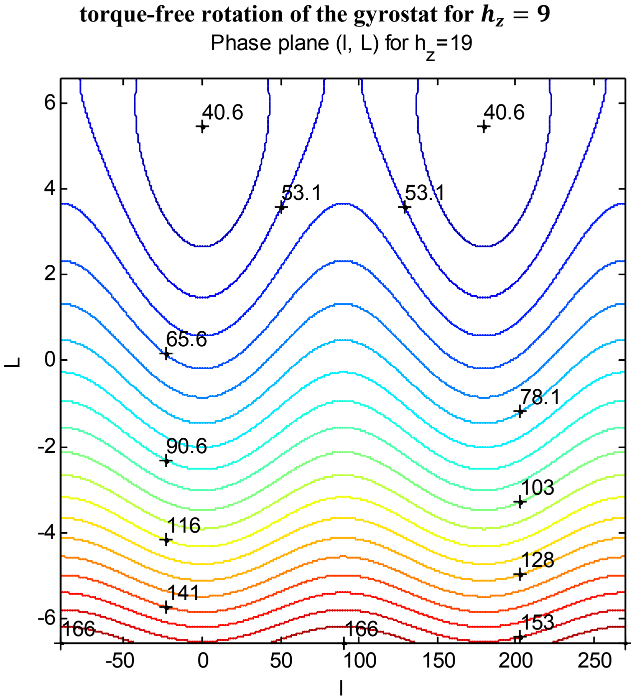

Figure 5. Isoenergetic curves in the phase plane (*l*,*L*) for the torque-free rotation of the gyrostat for $h_z = 19$

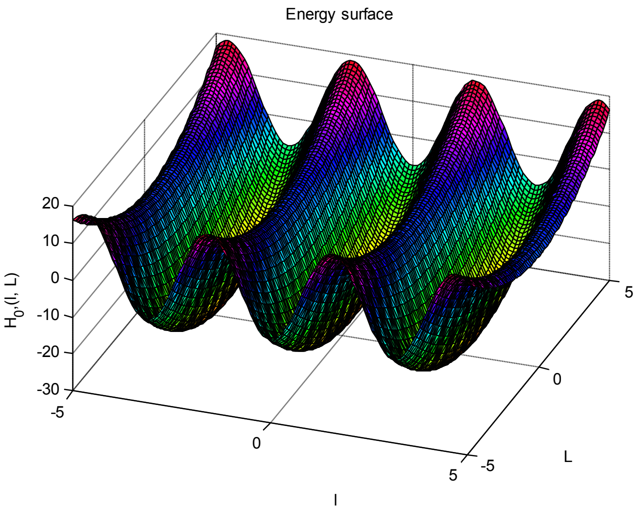

Figure 6. Energy surface for h_z=0

and the eigenvalues of this matrix are equal to zero.The results (using MATLAB) are plotted on phase plane (l, L) for the torque-free rotation of the gyrostat. Figures 3, 4 and 5 show isoenergetic curves in the phase rectangle (l, L) of the torque-free rotation of the gyrostat for the different angular momentum. For the torque-free rotation of the gyrostat, there are two saddle points for the first equilibrium at $l = 0$ and $l = \pi$. Figures 3 and 4 show phase plane for small angular momentum, $_z = 1.9$ and $_z = 9$, respectively. Can we see that, Homoclinic and heteroclinic trajectories are presented. For small relative angular momentum $_z$, that is, $0 < \; _z \leq (1 \quad C/B)G$, three equilibria coexist and same isoenergetic phase curves are simple closed curves around the equilibrium points.

Thus this equilibrium is stable; the gyrostat “librates” in the plane Oxy. As angular momentum is increased we have a new situation (see figure 5). So the number of equilibria changes as angular momentum is varied.

For $_z = 0$, the Hamiltonian (12) become

$$H_0 = \frac{1}{2}\left(\frac{sin^2 l}{A} + \frac{cos^2 l}{B}\right)(G^2 \quad L^2) + \frac{L^2}{2C} \tag{30}$$

Figure 6 show the Energy surface for the torque-free rotation of the gyrostat for h_z=0.

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