IJIRR

International Journal of Information Research and Review
Vol. 09, Issue, 02, pp.7395-7405, February, 2022

RESEARCH ARTICLE

A NEW DISTANCE MEASURE ON PYTHAGOREAN FUZZY SETS AND ITS APPLICATION TO DECISION MAKING

***Shengda Tao and Chunyong Wang**

School of Education and Music, Hezhou University, Hezhou 542899, China

ARTICLE INFO

Article History:
Received 26th November, 2021
Received in revised form
19th December, 2021
Accepted 20th January, 2022
Published online 28th February, 2022

Keywords:
Pythagorean Fuzzy set; Pythagorean fuzzy Number; Distance Measure; Multipleattribute Decision Making.

ABSTRACT

Based on integral and area, we construct a new distance for Pythagorean fuzzy set. Compared with Euclidean distance which may fail in some cases, the new distance has its advantages that can be seen by a numerical example. In addition, basic properties of the new distance are discussed. Moreover, we propose a modified TOPSIS approach based on this newly proposed distance measure. Finally, a practical example is presented to testify the efficiency and application of this new approach.

INTRODUCTION

Information plays a very important role in our daily life. It can maintain and improve the core competition ability. In order to easily gather information and finely describe information, many data structure have been proposed. For example, in order to collect distinct information, classic set proposed by Cantor is suitable. When we feel difficult to describe something with "yes" or "no", fuzzy set (FS) introduced by Zadeh (1) may be more applicable. Atanassov proposed the notion of intuitionistic fuzzy set (IFS for short)(2), which extend representational capability of FS. In IFS, every element x is equipped with membership degree $\mu(x)$ as well as non-membership degree $\nu(x)$, satisfying $\mu(x)+\nu(x)\leq 1$. Yager used the Pythagorean negation $Neg(a)=\sqrt{1-a^2}$ to replace the logical negation's linear form $Neg(a)=1-a$, and modified IFS to Pythagorean fuzzy set (10)(PFS), with $\mu^2(x)+\nu^2(x)\leq 1$. Since then, many researchers have begun using PFS to treat imprecision and uncertainty. They constructed Euclidean distance and different score functions to compare two Pythagorean fuzzy numbers (PFNs) and developed various multi-attribute decision making (MADM) methods with PFS. For example, Agheli (4) proposed a new method for calculating Pythagorean similar measure for two PFNs by using T-norm and S-norm. Rodriguez (9) developed a specification-assessment- compliance approach to obtain a transparent multi-criteria decision-making method. In (6), Li et al proposed a new similarity measures of PFSs based on the arc distance on sphere from the geometric perspective. By arc-length, Wan (7) proposed a relative closeness degree to rank PFNs. Inspired by the distance between intuitionistic fuzzy sets, the normalized Hamming distance (5) was defined as

$$d_P(A,B)=\frac{1}{n}\sum_{i=1}^{n}\left(\left|\mu_A(x_i)-\mu_B(x_i)\right|+\left|\eta_A(x_i)-\eta_B(x_i)\right|+\left|\nu_A(x_i)-\nu_B(x_i)\right|\right).$$

Zhang and Xu (11) constructed modified Euclidean distance between PFNs as below

Corresponding Author:** ***Shengda Tao **,**
School of Education and Music, Hezhou University, Hezhou 542899, China.

$$d(\beta_1,\beta_2)=\frac{1}{2}\left(\left|\left(\mu_{\beta_1}\right)^2-\left(\mu_{\beta_2}\right)^2\right|+\left|\left(\nu_{\beta_1}\right)^2-\left(\nu_{\beta_2}\right)^2\right|+\left|\left(\pi_{\beta_1}\right)^2-\left(\pi_{\beta_2}\right)^2\right|\right).$$

Nevertheless, most researchers were used to Euclidean distance between PFNs as follows (8).

$$d_E(\beta_1,\beta_2)=\sqrt{\frac{1}{2}\left\{\left[\left(\mu_{\beta_1}\right)^2-\left(\mu_{\beta_2}\right)^2\right]^2+\left[\left(\nu_{\beta_1}\right)^2-\left(\nu_{\beta_2}\right)^2\right]^2+\left[\left(\pi_{\beta_1}\right)^2-\left(\pi_{\beta_2}\right)^2\right]^2\right\}}$$

which can be seen as a natural generation of classic Euclidean distance. But all these distances above may fail in some special cases. For example, we can choose three points $P(0.2,0.7)$, $Q(0.2,0.1)$ and $P^+(1,0)$, then by simple computation we can see that $d_E(P,P^+)=d_E(Q,P^+)=\frac{\sqrt{5809}}{100}$. On the other hand, as $\mu_P=\mu_Q$, $\nu_P>\nu_Q$, we get $P\prec Q$, which indicates that $d_E(P,P^+)>d_E(Q,P^+)$. This is clearly a contradiction. Certainly, in this case, we can't compare P with Q by Euclidean distance. To overcome above-mentioned shortcoming, we will construct a new distance on Pythagorean fuzzy set. As distance measure is a basic notion for PFSs, it can derive many other notions such as ideal solution. As a result, it is very meaningful to improve distance measure for PFSs. The rest of this article is arranged as follows. In next section, we recap brief basics related to PFS. A new distance measure on PFS is introduced in Section 3.Some properties of the newly proposed distance measure are also discussed in Section 3.In Section 4, a comparison is made between new distance with existing ones. In Section 5, we propose a modified TOPSIS method for MADM by our new distance under PFS environment. A practical example is also demonstrated to show the effectiveness of this developed method in Section 6. In Section 7, we give the conclusion and some remarks.

Preliminaries: Let us briefly recap some basic concepts of PFS. In 1965, FS was proposed by Zadeh (1) as the following.

Definition 2.1.(1). Suppose X be a space of discourse, then the fuzzy set can be defined as: $A=\{\langle x,\mu_A(x)\rangle | x\in X\}$ which is described by a membership $\mu_A: X\to[0,1]$, where $\mu_A(x)$ represents degree of membership for element x to set A.
Atanassov (2) generalized FS to IFS as below.

Definition 2.2.(2). An IFS in X is defined by

$$A=\{\langle x,\mu_A(x),\nu_A(x)\rangle | x\in X\}$$

which is described by membership function $\mu_A: X\to[0,1]$ as well as non-membership $\nu_A: X\to[0,1]$, with the constraint $0\le\mu_A(x)+\nu_A(x)\le 1, \forall x\in X$, where $\mu_A(x)$ and $\nu_A(x)$ denote degree of membership and degree of non-membership for element x to set A, respectively. We can denote $(\mu_A(x),\nu_A(x))$ as an intuitionistic fuzzy number (IFN). If $\pi_A(x)=1-\mu_A(x)-\nu_A(x)=0, \forall x\in X$, then IFS will degenerate into FS. Yager changed the logical negation's linear form $Neg(a)=1-a$ in IFS to Pythagorean negation $Neg(a)=\sqrt{1-a^2}$, and introduced a non-standard fuzzy set named Pythagorean fuzzy set.

Definition2.3.(11). Let X be a space of discourse. The PFS P can be represented as

$$P=\{\langle x,P(\mu_P(x),\nu_P(x))\rangle | x\in X, 0\le\mu_P^2(x)+\nu_P^2(x)\le 1\}$$

where, $\mu_P(x)$ and $\nu_P(x)$ denote degree of membership and degree of non-membership for element x to set P, respectively. For a PFS P and $x\in X$, $\pi_P(x)=\sqrt{1-\mu_P^2(x)-\nu_P^2(x)}$ is called Pythagorean index of x to P. $P(\mu_P(x),\nu_P(x))$ can be called a PFN and denoted by $\beta=P(\mu_\beta,\nu_\beta)$ for short, where $\mu_\beta,\nu_\beta\in[0,1]$, $\mu_\beta^2(x)+\nu_\beta^2(x)\le 1$ and $\pi_\beta=\sqrt{1-\mu_\beta^2-\nu_\beta^2}$.

Formally speaking, IFN and PFN can be differed in their respective constraints. It can be easily seen that a PFN $P(\mu_\beta,\nu_\beta)$ will degenerate into an IFN if $\mu_\beta(x)+\nu_\beta(x)\le 1$.

A new distance measure on PFS and its basic properties: Here, we'll construct a novel distance measure on PFS. As shown in

Figure 1, there are two PFNs P and Q. $P=P(\mu_P,\nu_P)$ and $Q=Q(\mu_Q,\nu_Q)$ represents two PFNs, respectively. In what follows, we will consider how to measure the distance between P and Q.

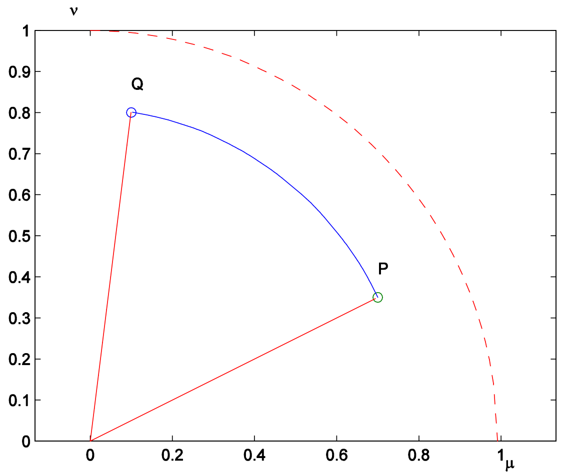

Figure 1. An intuitive explanation of new distance measure on PFS

For convenience, we convert the Cartesian coordinates above to polar coordinates. According to the definition of trigonometric function

$$r=\sqrt{1-x^2-y^2},0\le r\le 1,$$

$$\theta=\arctan\frac{y}{x},0\le\theta\le\frac{\pi}{2}.$$

Then, we smoothly get $P=P(r_P,\theta_P)$, $Q=Q(r_Q,\theta_Q)$.We will soon see the benefits of doing so. What is the distance between P and Q? Since each point in the region corresponds to a PFN, wecan naturally turn to compute the area which is composed by these points. Here comes another question: how to determine the area of domain $\overline{OPQ}$ marked in Figure 1?A compelling hypothesis is that r_M and θ_M change evenly as time goes by. The simplest model is: r_M and θ_M are linear functions of time. Thus, r_M is a linear function of θ_M.This makes us believe that the curve $\overline{PQ}$ can be described by $r=r(\theta)=k\theta+m$, where, k and m are unknown coefficients waiting us to determinate by other conditions. Luckily, we have already known the starting point $P=P(r_P,\theta_P)$ and end point $Q=Q(r_Q,\theta_Q)$ of the curve. Therefore, we acquire the following constrains:

$$r_P=k\theta_P+m,$$

$$r_Q=k\theta_Q+m.$$

Solving these equations, we obtain

$$k=\frac{r_Q-r_P}{\theta_Q-\theta_P},$$

$$m=\frac{r_P\theta_Q-r_Q\theta_P}{\theta_Q-\theta_P}.$$

We are pleased to describe $\overline{PQ}$ as

$$r = r(\theta) = \frac{r_Q - r_P}{\theta_Q - \theta_P}\theta + \frac{r_P\theta_Q - r_Q\theta_P}{\theta_Q - \theta_P} = r_P + \frac{r_Q - r_P}{\theta_Q - \theta_P}(\theta - \theta_P).$$

In fact, we can gain the equation above by the point oblique equation of a straight line. Next, wecan easily calculate the area $\overline{OPQ}$ by integral:

$$S_{\overline{OPQ}} = \frac{1}{2}\left|\int_{\theta_P}^{\theta_Q} r^2(\theta)d\theta\right|.$$

In order to standardize the distance $d(P,Q) \in [0,1]$, we divide $S_{\overline{OPQ}}$ by its maximum $\frac{\pi}{4}$,then wecome to the following concept.

Definition 3.1. Let $P = P(r_P, \theta_P)$ and $Q = Q(r_Q, \theta_Q)$ be PFNs, which are expressed by polar coordinates. The distance between P and Q can be described by

$$d(P,Q) = \frac{2}{\pi}\left|\int_{\theta_P}^{\theta_Q} r^2(\theta)d\theta\right|.$$

where, $r(\theta) = r_P + \frac{r_Q - r_P}{\theta_Q - \theta_P}(\theta - \theta_P)$.

In what follows, we will discuss several properties of the proposed distance measure $d(P,Q)$ on PFS. Apparently, $0 \le d(P,Q) \le 1$, so the following proposition holds.

Theorem 3.2 (Boundness) Let $P = P(r_P, \theta_P)$ and $Q = Q(r_Q, \theta_Q)$ be PFNs expressed by polar coordinates, then the distance between P and Q ranges from 0to 1: $0 \le d(P,Q) \le 1$.Furthermore, $d(P,Q) = 0$ if and only if $\theta_P = \theta_Q$.

Proof. By definition3.1 and $0 \le r \le 1$, $|\theta_P - \theta_Q| \le \frac{\pi}{2}$, we have

$$d(P,Q) = \frac{2}{\pi}\left|\int_{\theta_P}^{\theta_Q} r^2(\theta)d\theta\right| \le \frac{2}{\pi}\left|\int_{\theta_P}^{\theta_Q} d\theta\right| = \frac{2}{\pi}|\theta_P - \theta_Q| \le \frac{2}{\pi} \times \frac{\pi}{2} = 1.$$

If $\theta_P \ne \theta_Q$, as r is continuous and not always equal to 0, we can infer: $\exists \theta_0 \in (\theta_P, \theta_Q)$, $\delta > 0$, $r_0 > 0$ such that $r(\theta) > r_0$ for $\forall \theta \in U(\theta_0, \delta) \subseteq (\theta_P, \theta_Q)$. As a result,

$$d(P,Q) = \frac{2}{\pi}\left|\int_{\theta_P}^{\theta_Q} r^2(\theta)d\theta\right| \ge \frac{2}{\pi}\left|\int_{\theta_0-\delta}^{\theta_0+\delta} r_0^2 d\theta\right| = \frac{2}{\pi} \times r_0 \times \delta > 0.$$

Which completes this proof of Theorem 3.2.

Theorem 3.3.(Symmetry) Let P and Q be PFNs, then we acquire $d(P,Q) = d(Q,P)$.

Proof. The proof is straightforward due to we have an absolute value operation in Definition 3.1.

Theorem 3.4. (Triangle inequality)Let $P = P(r_P, \theta_P)$, $Q = Q(r_Q, \theta_Q)$ and $R = R(r_R, \theta_R)$ be PFNs expressed by polar coordinates. If $\theta_R \le \theta_Q \le \theta_P$ and $\frac{r_Q - r_R}{\theta_Q - \theta_R} \ge \frac{r_P - r_R}{\theta_P - \theta_R}$, then we obtain $d(R,P) = d(R,Q) + d(Q,P)$.

Proof. In order toga in an intuitive understanding, we put these points *P, Q, R* in Figure 2.

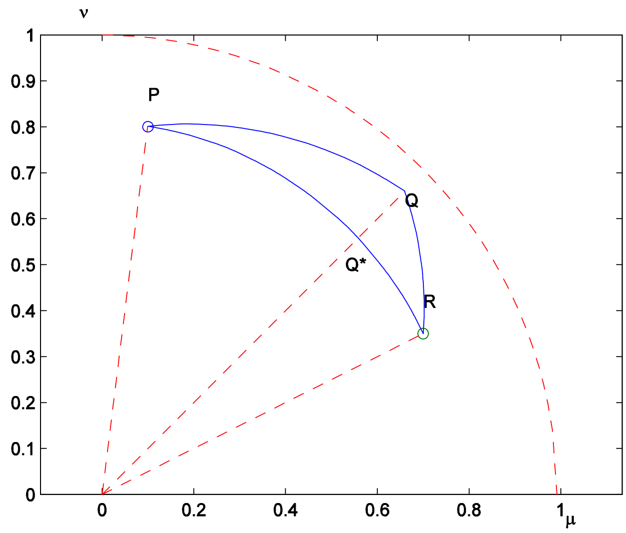

Figure 2. Triangle inequality under certain conditions

As $\frac{r_Q - r_R}{\theta_Q - \theta_R} \geq \frac{r_P - r_R}{\theta_P - \theta_R}$, we find that for $\theta_R \leq \theta_Q \leq \theta_P$, $r_{PR} \leq r_{QR}$.

As seen in Figure 2, we choose Q^* to denote the intersection point of two curves: $\overline{PR}$ and $\overline{OQ}$. As a comparison object, we have $Q^*\left(r_{Q^*}, \theta_{Q^*}\right)$ with $r_{Q^*} \leq r_Q$, $\theta_{Q^*} \leq \theta_Q$. Similarly, we compare curves

$$\overline{Q^*P}: r_{Q^*P}(\theta) = r_{Q^*} + \frac{r_P - r_{Q^*}}{\theta_P - \theta_{Q^*}}\left(\theta - \theta_{Q^*}\right), \theta_{Q^*} \leq \theta \leq \theta_P,$$

$$\overline{QP}: r_{QP}(\theta) = r_Q + \frac{r_P - r_Q}{\theta_P - \theta_Q}\left(\theta - \theta_Q\right), \theta_Q \leq \theta \leq \theta_P.$$

When we compare curves

$$\overline{PR}: r_{PR}(\theta) = r_R + \frac{r_P - r_R}{\theta_P - \theta_R}\left(\theta - \theta_R\right), \theta_R \leq \theta \leq \theta_P,$$

$$\overline{QR}: r_{QR}(\theta) = r_R + \frac{r_Q - r_R}{\theta_Q - \theta_R}\left(\theta - \theta_R\right), \theta_R \leq \theta \leq \theta_Q.$$

In order to compare r_{Q^*P} and r_{QP}, we compute

$$\begin{aligned} r_{QP}(\theta) - r_{Q^*P}(\theta) &= r_Q - r_{Q^*} + \frac{\theta - \theta_Q}{\theta_P - \theta_Q}\left(r_{Q^*} - r_Q\right) \\ &= \left(r_Q - r_{Q^*}\right)\left(1 - \frac{\theta - \theta_Q}{\theta_P - \theta_Q}\right) \\ &= \left(r_Q - r_{Q^*}\right)\frac{\theta_P - \theta}{\theta_P - \theta_Q}. \end{aligned}$$

As we know, $\theta_R \leq \theta_Q \leq \theta_P$ and $r_{Q^*} \leq r_Q$, we get $r_{QP}(\theta) - r_{Q^*P}(\theta) \geq 0$, which indicates that $r_{QP}(\theta) \geq r_{Q^*P}(\theta)$ for $\forall \theta \in [\theta_Q, \theta_P]$. Therefore, by Definition 3.1, we have

$$
\begin{aligned}
d(R,P) &= \frac{2}{\pi}\left|\int_{\theta_R}^{\theta_P} r^2_{\overline{RQ^*P}}(\theta)\,d\theta\right| \\
&= \frac{2}{\pi}\left|\int_{\theta_R}^{\theta_{Q^*}} r^2_{\overline{RQ^*P}}(\theta)\,d\theta\right| + \frac{2}{\pi}\left|\int_{\theta_{Q^*}}^{\theta_P} r^2_{\overline{RQ^*P}}(\theta)\,d\theta\right| \\
&\le \frac{2}{\pi}\left|\int_{\theta_R}^{\theta_Q} r^2_{\overline{RQ}}(\theta)\,d\theta\right| + \frac{2}{\pi}\left|\int_{\theta_Q}^{\theta_P} r^2_{\overline{QP}}(\theta)\,d\theta\right| \\
&= d(R,Q) + d(Q,P).
\end{aligned}
$$

Which completes the proof of Theorem3.4.

Theorem3.5. Let $P(r_P,\theta_P)$ be a PFN expressed by polar coordinates. Set $P^-\left(1,\frac{\pi}{2}\right)$ and $P^+(1,0)$ be the negative ideal PFN and the positive ideal PFN, respectively. Then we have

$$
d(P,P^+) = \frac{2\theta_P}{3\pi}\left(r_P^2 + r_P + 1\right),
$$

$$
d(P,P^-) = \frac{\pi - 2\theta_P}{3\pi}\left(r_P^2 + r_P + 1\right).
$$

Proof. To verify Theorem3.5, we only need to calculate the definite integral given in Difinition3.1.We only calculate $d(P,P^+)$ as follows.

$$
\begin{aligned}
d(P,P^+) &= \frac{2}{\pi}\left|\int_0^{\theta_P} r^2(\theta)\,d\theta\right| \\
&= \frac{2}{\pi}\left|\int_0^{\theta_P}\left(1+\frac{1-r_P}{\theta-\theta_P}\theta\right)^2 d\theta\right| \\
&= \frac{2}{\pi}\left[\theta - \frac{1-r_P}{\theta_P}\theta^2 + \frac{1}{3}\left(\frac{1-r_P}{\theta_P}\right)^2\theta^3\right]_{\theta=0}^{\theta=\theta_P} \\
&= \frac{2\theta_P}{3\pi}\left(r_P^2 + r_P + 1\right)
\end{aligned}
$$

As the calculation about $d(P,P^+)$ is similar to $d(P,P^+)$, we omit it here.

Comparison with the existing distance measure: In this section, we will compare our new distance measure with existing Euclidean distance on PFS. In (8), Ren introduced a Euclidean distance of PFNs as the following.

Definition 4.1.(8).Let $\beta_1 = (\mu_{\beta_1}, \nu_{\beta_1}), \beta_2 = (\mu_{\beta_2}, \nu_{\beta_2})$ be two PFNs which are expressed by Cartesian coordinates. Then the Euclidean distance between β_1 and β_2 is:

$$
d_E(\beta_1,\beta_1) = \sqrt{\frac{1}{2}\left\{\left[(\mu_{\beta_1})^2 - (\mu_{\beta_2})^2\right]^2 + \left[(\nu_{\beta_1})^2 - (\nu_{\beta_2})^2\right]^2 + \left[(\pi_{\beta_1})^2 - (\pi_{\beta_2})^2\right]^2\right\}}.
$$

Remark 4.2. Ren's Euclidean distance is a generation of distance on IFS. The d_E reflects the difference of membership, non-membership and hesitant degree between β_1 and β_2. Nevertheless, it is somewhat subjective for giving the same weight $\frac{1}{2}$ to

$$
\left[(\mu_{\beta_1})^2 - (\mu_{\beta_2})^2\right]^2,\ \left[(\nu_{\beta_1})^2 - (\nu_{\beta_2})^2\right]^2,\ \left[(\pi_{\beta_1})^2 - (\pi_{\beta_2})^2\right]^2.
$$

The following example will show the shortcomings of d_E.

Example 4.3. We consider three points $P(0.2,0.7), Q(0.2,0.1)$ and $P^{+}(1,0)$, which are represented by Cartesian coordinates. By Ren's Definition4.1, we can get

$$
\begin{aligned}
d_E\left(P,P^{+}\right) &= \sqrt{\frac{1}{2}\left\{\left[\left(\mu_{\beta_1}\right)^2-\left(\mu_{\beta_2}\right)^2\right]^2+\left[\left(\nu_{\beta_1}\right)^2-\left(\nu_{\beta_2}\right)^2\right]^2+\left[\left(\pi_{\beta_1}\right)^2-\left(\pi_{\beta_2}\right)^2\right]^2\right\}}\\
&= \sqrt{\frac{1}{2}\left\{\left[(0.2)^2-(1)^2\right]^2+\left[(0.7)^2-(0)^2\right]^2+\left[(1-0.2-0.7)^2-(1-1-0)^2\right]^2\right\}}\\
&= \sqrt{5809}/100,\\
d_E\left(Q,P^{+}\right) &= \sqrt{\frac{1}{2}\left\{\left[\left(\mu_{\beta_1}\right)^2-\left(\mu_{\beta_2}\right)^2\right]^2+\left[\left(\nu_{\beta_1}\right)^2-\left(\nu_{\beta_2}\right)^2\right]^2+\left[\left(\pi_{\beta_1}\right)^2-\left(\pi_{\beta_2}\right)^2\right]^2\right\}}\\
&= \sqrt{\frac{1}{2}\left\{\left[(0.2)^2-(1)^2\right]^2+\left[(0.1)^2-(0)^2\right]^2+\left[(1-0.2-0.1)^2-(1-1-0)^2\right]^2\right\}}\\
&= \sqrt{5809}/100.
\end{aligned}
$$

As we can see $d_E\left(P,P^{+}\right)=d_E\left(Q,P^{+}\right)$, which means the distance from the positive ideal PFN P^{+} to P is the same as the distance from P^{+} to Q.However, from $\mu_P=\mu_Q=0.2<1=\mu_{P^{+}}$ and $\nu_P=0.7>0.1=\nu_Q>0=\nu_{P^{+}}$, we can infer that the ranking should be $P \prec Q \prec P^{+}$.As a result, the ranking of Euclidean distance should be $d_E\left(P,P^{+}\right)>d_E\left(Q,P^{+}\right)$. For this example, the Euclidean distance is not reasonable enough.

Let us measure the above distance by our newly defined distance d. We should convert the Cartesian coordinates into polar coordinates as $r_P=\sqrt{0.2^2+0.7^2}=\frac{\sqrt{53}}{10}, \theta_P=\arctan\frac{7}{2}$.Similarly, we get $r_Q=\sqrt{0.2^2+0.1^2}=\frac{1}{2\sqrt{5}}, \theta_Q=\arctan\frac{1}{2}$ and $r_{P^{+}}=1$, $\theta_{P^{+}}=0$.

Put these values into the Theorem3.5, we obtain

$$d\left(P,P^{+}\right)=\frac{2\theta_P}{3\pi}\left(r_P^2+r_P+1\right)\approx 0.619319,$$

$$d\left(Q,P^{+}\right)=\frac{2\theta_P}{3\pi}\left(r_P^2+r_P+1\right)\approx 0.125309.$$

Here, $d\left(P,P^{+}\right)>d\left(Q,P^{+}\right)$ indicates that P is farther from the positive ideal PFN than Q which matches $P \prec Q$better. In this case, our newly defined distance performs better and it has advantages. The example above is not an isolated case. In fact, even when we fixed point $P(0.2,0.7)$, the classic Euclidean distance can fail at in finite number of points except for $Q(0.2,0.1)$. Denoting the moving point $N(\mu,\nu)$, now we write the distance equation:

$$d_E\left(P,P^{+}\right)>d_E\left(N,P^{+}\right).$$

Then, substitute $P(0.2,0.7)$ and $\pi=1-\mu-\nu$ into the equation above, we get

$$\sqrt{\frac{1}{2}\left\{\left[(\mu)^2-(1)^2\right]^2+\left[(\nu)^2-(0)^2\right]^2+\left[(1-\mu-\nu)^2-(0)^2\right]^2\right\}}=\frac{\sqrt{5809}}{100}.$$

This means, all the points which has the same Euclidean distance from P^{+} with P form the curve:

$$\frac{\sqrt{\left(\mu^2-1\right)^2+\left(1-\mu-\nu\right)^2+\nu^4}}{\sqrt{2}}=\frac{\sqrt{5809}}{100}$$, please see Figure 3.

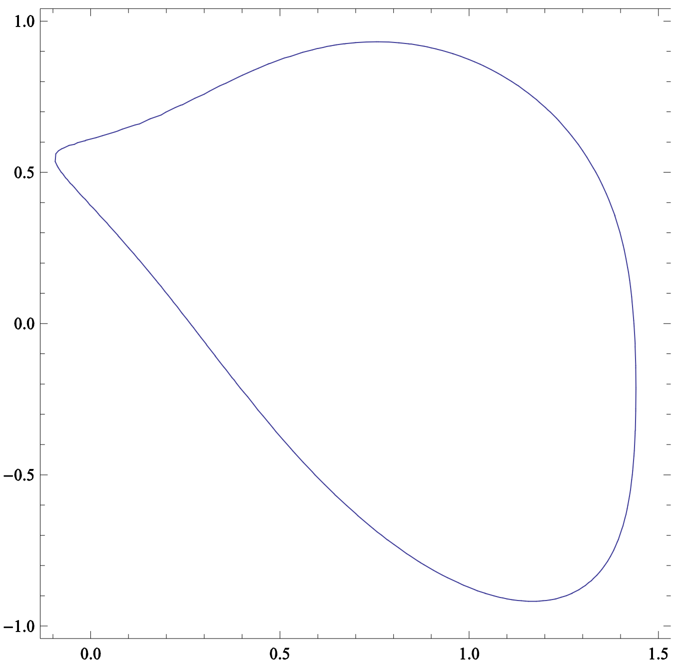

Figure 3. All the points which has the same Euclidean distance from P^+ with P form a curve

A modified TOPSIS method for MADM with Pythagorean fuzzy sets: To solve above-mentioned MCDM with PFS effectively, we will propose a modified TOPSIS approach based on the our newly introduced distance measure d on PFSs. According to the concise principle, TOPSIS method assumes that the optimal alternative should be the nearest to positive ideal marked by x^+ and the furthest from negative ideal denoted by x^-. A MADM with PFSs can be denoted by a decision matrix whose elements indicate the evaluation assess values of each alternatives to each attribute. Let $X=\{x_1,x_2,\cdots,x_m\}$ be a set containing m alternatives, $C=\{C_1,C_2,\cdots,C_n\}$ be a set of n attributes, $W=\{w_1,w_2,\cdots,w_n\}$ be the corresponding weight vector for all attributes, satisfying $0\le w_i\le 1$ and $\sum_{i=1}^{n} w_i=1$. If the decision makers give evaluation value for alternative x_i under attribute C_j with anonymity, the value can be expressed as a Pythagorean fuzzy number $C_j(x_i)=P(r_{ij},\theta_{ij})$. Then the decision matrix $R=\left(C_j(x_i)\right)_{m\times n}$ is a Pythagorean hesitant fuzzy decision matrix. Thus, MCDM with PFSs can be expressed by following matrix:

$$R=\left(C_j(x_i)\right)_{m\times n}=\begin{pmatrix} P(r_{11},\theta_{11}) & P(r_{12},\theta_{12}) & \cdots & P(r_{1n},\theta_{1n}) \\ P(r_{21},\theta_{21}) & P(r_{22},\theta_{22}) & \cdots & P(r_{2n},\theta_{2n}) \\ \vdots & \vdots & \vdots & \vdots \\ P(r_{m1},\theta_{m1}) & P(r_{m2},\theta_{m2}) & \cdots & P(r_{mn},\theta_{mn}) \end{pmatrix}.$$

In what follows, we propose a modified TOPSIS method for MADM problem with Pythagorean fuzzy sets.

Step 1.We construct decision matrix $R=\left(C_j(x_i)\right)_{m\times n}$, whose element $C_j(x_i)=P(r_{ij},\theta_{ij})$ carries information expressed by PFNs.

Step 2.Seek negative ideal x^- and positive ideal x^+ (11) as:

$$x^-=\left\{\left\langle C_j,\min_i\left\langle s\left(C_j(x_i)\right)\right\rangle\right\rangle \middle| j=1,2,\cdots,n\right\},$$

$$x^+=\left\{\left\langle C_j,\max_i\left\langle s\left(C_j(x_i)\right)\right\rangle\right\rangle \middle| j=1,2,\cdots,n\right\}.$$

Step 3. Compute the distance between x_i and x^+ denoted by $d(x_i,x^+)$ according to Definition 3.1.Analogously, we can acquire the distance between x_i and x^- denoted by $d(x_i,x^-),1\le i\le m$.

Step 4. Reckon the revised closeness $\varsigma(x_i)$ (3) for the alternative $x_i,1\le i\le m$ as:

$$\varsigma(x_i)=\frac{d(x_i,x^-)}{\max_{1\le i\le m} d(x_i,x^-)}-\frac{d(x_i,x^+)}{\max_{1\le i\le m} d(x_i,x^+)}.$$

Step 5. Rank all the alternatives $x_i\left(1\le i\le m\right)$ by the revised closeness $\varsigma\left(x_i\right)$ and select the best one(s) x^* such that $\varsigma\left(x^*\right)=\max_{1\le i\le m}\varsigma\left(x_i\right)$.

Step 6. End.

Illustration example: Here, we will consider an actual MADM about the evaluation of service among several domestic airlines (11). For ease of comparison, we use data in(11) without making any changes. In order to evaluate four domestic airlines marked by x_1,x_2,x_3,x_4, four criteria are choosing as: C_1,C_2,C_3,C_4, which correspond to Booking and ticketing, Check-in and boarding process, Cabin service, Responsiveness, respectively. To know the details of this evaluation, please refer to (11).The weight vector $w=(0.15,0.25,0.35,0.25)$ for attributes is provided by experts in advance. For each airline, the assessment values under each attributes are expressed by PFNs shown in Table 1.

Table 1. Pythagorean Fuzzy Decision Matrix $R_{4\times4}$.

	C_1	C_2	C_3	C_4
x_1	P(0.9,0.3)	P(0.7,0.6)	P(0.5,0.8)	P(0.6,0.3)
x_2	P(0.4,0.7)	P(0.9,0.2)	P(0.8,0.1)	P(0.5,0.3)
x_3	P(0.8,0.4)	P(0.7,0.5)	P(0.6,0.2)	P(0.7,0.4)
x_4	P(0.7,0.2)	P(0.8,0.2)	P(0.8,0.4)	P(0.6,0.6)

Take $C_1\left(x_1\right)=P\left(0.9,0.3\right)$ for example, it means that degree of alternative x_1 satisfies attribute C_1 is 0.9, while degree of alternative x_1 dissatisfies attribute C_1 is 0.3.Other PFNs in Table 1have similar meanings.

In the following, we adopt the modified TOPSIS method to solve the above mentioned decision making problem. As the decision matrix $R=\left(C_j\left(x_i\right)\right)_{m\times n}$ has already been constructed, webegin from step 2 in Section 7 to seek negative ideal x^- and positive ideal x^+ (11)as:

$$x^-=\left\{\left\langle C_j,\min_i\left\langle s\left(C_j\left(x_i\right)\right)\right\rangle\right\rangle\middle|j=1,2,3,4\right\}=\left\{P\left(0.4,0.7\right),P\left(0.7,0.6\right),P\left(0.5,0.8\right),P\left(0.6,0.6\right)\right\},$$

$$x^+=\left\{\left\langle C_j,\max_i\left\langle s\left(C_j\left(x_i\right)\right)\right\rangle\right\rangle\middle|j=1,2,3,4\right\}=\left\{P\left(0.9,0.3\right),P\left(0.9,0.2\right),P\left(0.8,0.1\right),P\left(0.7,0.4\right)\right\},$$

Step 3. Calculate the distance between x_i and x^+ denoted by $d\left(x_i,x^+\right)$ according to Definition 3.1. Analogously, we can acquire the distance between x_i and x^- denoted by $d\left(x_i,x^-\right)$. Here, we only take $d\left(x_1,x^+\right)$ for example.

$$d\left(x_1,x^+\right)=\sum_{j=1}^{4}w_jd\left(C_j\left(x_1\right),C_j\left(x^+\right)\right)$$

$$=w_1d\left(C_1\left(x_1\right),C_1\left(x^+\right)\right)+w_2d\left(C_2\left(x_1\right),C_2\left(x^+\right)\right)+w_3d\left(C_3\left(x_1\right),C_3\left(x^+\right)\right)+w_4d\left(C_4\left(x_1\right),C_4\left(x^+\right)\right)$$

$$=0.15\times\frac{2}{\pi}\left|\int_{\theta_{P(0.9,0.3)}}^{\theta_{Q(0.9,0.3)}}r^2\left(\theta\right)d\theta\right|+0.25\times\frac{2}{\pi}\left|\int_{\theta_{P(0.7,0.6)}}^{\theta_{Q(0.9,0.2)}}r^2\left(\theta\right)d\theta\right|+0.35\times\frac{2}{\pi}\left|\int_{\theta_{P(0.5,0.8)}}^{\theta_{Q(0.8,0.1)}}r^2\left(\theta\right)d\theta\right|+0.25\times\frac{2}{\pi}\left|\int_{\theta_{P(0.6,0.3)}}^{\theta_{Q(0.7,0.4)}}r^2\left(\theta\right)d\theta\right|$$

$$=0.15\times0+0.25\times0.31831+0.35\times0.433445+0.25\times0.0193243=0.236114.$$

Similarly, we can acquire the results, please see Table 2.

Table 2: The distance between x_i and x^+ as well as x^-.

	$d\left(x_i,x^+\right)$	$d\left(x_i,x^-\right)$	$\varsigma\left(x_i\right)$	Ranking
x_1	0.236114	0.0834692	-0.668101	4
x_2	0.0554298	0.25149	0.765241	1
x_3	0.0851514	0.177573	0.345447	3
x_4	0.0898563	0.203062	0.426872	2

Step 4. Reckon the revised closeness $\varsigma\left(x_i\right)$ (3) for all alternative $x_i, i=1,2,3,4$. Here, we take $\varsigma\left(x_1\right)$ for example

$$\begin{aligned}\varsigma\left(x_1\right) &= \frac{d(x_1,x^-)}{\max_{1\le i\le m} d(x_i,x^-)} - \frac{d(x_1,x^+)}{\max_{1\le i\le m} d(x_i,x^+)} \\ &= \frac{0.0834692}{0.25149} - \frac{0.236114}{0,236114} = -0.668101.\end{aligned}$$

Analogously, we can acquire other results which can be found in Table 2. Step 5.According to the revised closeness $\varsigma\left(x_i\right)$, all the alternatives $x_i\left(i=1,2,3,4\right)$ can be ranked as

$x_2 \succ x_4 \succ x_3 \succ x_1$.

So, the best alternative is x_2.

Step 6. End.

Yager (10) gave an effective method by the PFWA aggregation operator to solve MADM with PFS. In order to compare our method with Yager's method (10) and Zhang's method(11), we use these method to solve the illustration example above. As Zhang (11) has already solve the same illustration example by Yager's method and Zhang's method, we only list the results in Table 3.For the detailed computation, we can refer to literature (11).

Table 3. Compare our method with existing ones.

methods	Rankings
The proposed method	$x_2 \succ x_4 \succ x_3 \succ x_1$
Yager's method	$x_2 \succ x_4 \succ x_3 \succ x_1$
Zhang's method	$x_2 \succ x_3 \succ x_4 \succ x_1$

From Table 3, we can easily find that the rankings of potential alternatives acquired by three different methods are almost the same. The best alternative chosen by all these methods is always x_2, which indicates that the newly proposed method is valid.

Conclusion

In this paper, we construct a new distance on PFS, which is intuitive and interesting. Basing on this newly defined distance, we also develop a modified TOPSIS method for MADM with PFS. Finally, a practical example about evaluation of domestic airlines has been given to demonstrate its effectiveness and practicality. In the future, we will try to construct new score function on PFS and develop novel method to MADM under Pythagorean fuzzy environment.

Acknowledgments

This work was supported by the Project of improving the basic ability of Young and Middle-aged Teachers in Guangxi Universities (No. 2019KY0720) and National Social Science Foundation Projects (No. 18XTJ002).

REFERENCES

1. Zadeh L A, Fuzzy sets, Information and Control 8 (1965) 338-353.
2. Atanassov, K.T. Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87-96.
3. Hadi-Vencheh A, Mirjaberi M, Fuzzy inferior ratio method for multiple attribute decision making problems, Information Sciences 277 (2014) 263-272.
4. Agheli B, Firozja M A, Garg H, Similarity measure for Pythagorean fuzzy sets and application on multiplecriteria decision making, Journal of Statistics and Management Systems, DOI: 10.1080/09720510.2021.1891699.
5. Szmidt E, Kacprzyk J, Distances between intuitionistic fuzzy sets, Fuzzy Sets and Systems 114 (2000) 505-518.
6. J Li, L Wen, G Wei, J Wu, C Wei, New similarity and distance measures of Pythagorean fuzzy sets and its application to selection of advertising platforms, Journal of Intelligent and Fuzzy Systems 40 (2021) 1-17.
7. Wan S P, Jin Z, Wang F, A new ranking method for Pythagorean fuzzy numbers, International Conference on Intelligent Systems and Knowledge Engineering. IEEE(2018) 1-6.
8. Ren P, Xu Z, Gou X, Pythagorean fuzzy TODIM approach to multi-criteria decision making, Applied Soft Computing 42 (2016) 246-259.
9. A Rodriguez, A risk assessment method based on Pythagorean fuzzy set and artificial-neuron-like evaluation node, Journal of Intelligent and Fuzzy Systems 1 (2021) 1-12
10. Yager R R. Pythagorean membership grades in multicriteria decision making, IEEE Transactions on Fuzzy Systems 22 (2014) 958-965
11. Zhang X, Xu Z, Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets, International Journal of Intelligent Systems 29 (2015) 1061-1078.

****